Non-Hitchin Borel Anosov representations from surface groups to $\mathrm{SL}_{3k}\mathbb{R}$

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张峻铭 Junming Zhang

Chern Institute of Mathematics, Nankai University

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南開大學
Nankai University

Higher (rank) Teichmüller theory

Classical Teichmüller theory

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Moreover, $\mathcal{T}(S)$ is one of the two connected components which consist entirely of discrete and faithful representations. The other one is $\mathcal{T}(\overline{S})$, where $\overline{S}$ denotes the surface S with the opposite orientation.
NO other component contains a discrete and faithful representation.

Anosov Property of Fuchsian Representation

For a Fuchsian representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{R}$, we have

$$\log \frac{|\lambda_1(\rho(\gamma))|}{|\lambda_2(\rho(\gamma))|} = \ell_{\mathbb{H}^2/(\mathrm{im}(\rho))}(\gamma),$$

where λ_i denotes the i-th eigenvalue.

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and so on.

In general, the set of discrete and faithful representations is only closed and NOT open.

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- (Hitchin, 1992; Labourie, 2006; Fock–Goncharov, 2006) Hitchin components $\mathcal{T}_{\mathrm{Hit}}(S, G)$ for **real split** G, such as $\mathrm{SL}_n\mathbb{R}$, $\mathrm{SO}^0_{n,n+1}$, $\mathrm{Sp}_{2n}\mathbb{R}$ and so on;

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- (Burger–Iozzi–Wienhard, 2010) Maximal components $\mathcal{T}_{\max}(S, G)$ for **Hermitian** G, such as $\mathrm{SU}_{p,q}$, $\mathrm{SO}^0_{2,n}$ and so on.

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When $G = \mathrm{PSL}_2\mathbb{R}$, the above components coincide with the classical Teichmüller space.

Anosov representations

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The original definition is highly dynamical and we will use an interpreted definition thanks to

- Guichard–Wienhard;
- Kapovich–Leeb–Porti;
- Guéritaud–Guichard–Kassel–Wienhard;
- Bochi–Potrie–Sambarino;
- Kassel–Potrie;
- and many others.

Definition

Fix a hyperbolic structure X on S. A representation $\rho\colon \pi_1(S) \to \mathrm{SL}_n\mathbb{C}$ is P_k-**Anosov** if there exist constants $C_1, C_2 > 0$ such that

$$\log \frac{|\lambda_k(\rho(\gamma))|}{|\lambda_{k+1}(\rho(\gamma))|} \geqslant C_1 \cdot \ell_X(\gamma) - C_2, \forall \gamma \in \pi_1(S),$$

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Anosov $\Longrightarrow$ discrete + faithful.

Anosov property is open but not closed. However, all representations in the known higher Teichmüller spaces are Anosov.

Hitchin representation

A representation $\rho\colon \pi_1(S) \to \mathrm{SL}_n\mathbb{R}$ is called **Hitchin** if it is path-connected with $\tau_n \circ j$ for a Fuchsian representation j, where τ_n denotes the unique irreducible $\mathrm{SL}_2\mathbb{R} \to \mathrm{SL}_n\mathbb{R}$.

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- **True when $n = 2$;**
- **Trivially false when n is odd: $(\tau_{n-1} \circ j) \oplus \mathbf{1}$.**

Barbot representation

Let $n = 3$, $X := \mathbb{H}^2/\operatorname{im}(j)$. $\rho_0 = j \oplus \mathbf{1} \colon \pi_1(S) \to \mathrm{SL}_3\mathbb{R}$ is called a **Barbot representation**. It is NOT Hitchin (since Hitchin representations are irreducible).

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Theorem (Yao–Z., 2026+)

*There **exist** irreducible non-Hitchin Borel-Anosov representations from $\pi_1(S)$ to $\mathrm{SL}_{3k}\mathbb{R}$, in particular to $\mathrm{SL}_{6k}\mathbb{R}$, for every $k \geqslant 1$.*

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We need **Higgs bundles** to construct such deformation.

Non-Abelian Hodge correspondence

Higgs bundles

The Higgs bundle is a useful tool to study the higher Teichmüller space. It is usually used to give a parametrization of the higher Teichmüller space. We fix a complex structure on S such that it becomes a Riemann surface X. Let $\mathcal{K}_X$ be its canonical line bundle.

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Definition

A **($\mathrm{GL}(n,\mathbb{C})$-)Higgs bundle** over X is a pair $(\mathcal{E},\Phi)$ consisting of the following data:

- a holomorphic vector bundle $\mathcal{E}$ over X with $\mathrm{rank}(\mathcal{E})=n$;
- a holomorphic section $\Phi\in \mathrm{H}^0(X,\mathrm{End}(\mathcal{E})\otimes\mathcal{K}_X)$ called **Higgs field**.

The non-Abelian Hodge correspondence exhibit a real-analytic homeomorphism between the following moduli spaces:

$$\{\textbf{reductive representation } \rho\colon \pi_1(S) \to \mathrm{GL}(n,\mathbb{C})\}/\mathrm{GL}(n,\mathbb{C})$$

$$\text{associated bundle} \downarrow \quad \uparrow \text{holonomy}$$

$$\{\textbf{reductive flat vector bundle } (E \to S, \nabla) \text{ with } \mathrm{rank}(E) = n\}/\mathcal{G}$$

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If we equip additional structure on these objects, we can get the non-Abelian Hodge correspondence for general reductive Lie group G.

From Higgs bundle to flat bundle

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Theorem (Hitchin, 1987; Simpson, 1988)

*If $(\mathcal{E}, \Phi)$ is a polystable Higgs bundle with $\deg(\mathcal{E}) = 0$, then there exists an Hermitian metric h (**harmonic metric**) on $\mathcal{E}$ such that*

$$F(\nabla^h) + [\Phi \wedge \Phi^{*_h}] = 0, \tag{1}$$

where ∇^h is the Chern connection of the metric h, $F(\nabla^h)$ denotes its curvature form and Φ^{*_h} is the adjoint of Φ with respect to h. Moreover, if $(\mathcal{E}, \Phi)$ is stable, then such h is unique up to a constant scalar.

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If h solves (1), then

$$\nabla^h + \Phi + \Phi^{*_h}$$

gives a flat connection.

Deformation along Bronstein–Davalo slice

Fuchsian representation in Higgs bundle viewpoint

Let X be a Riemann surface structure on S. We fix a choice of $\mathcal{K}_X^{1/2}$. Through the non-Abelian Hodge correspondence,

$$\left(\mathcal{K}_X^{1/2} \oplus \mathcal{K}_X^{-1/2}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)$$

gives the Fuchsian representation j corresponding to the hyperbolic structure of X. Different choice of $\mathcal{K}_X^{1/2}$ gives different lifts from $\mathrm{PSL}_2\mathbb{R}$ to $\mathrm{SL}_2\mathbb{R}$.

Barbot representation and its deformation

The Barbot representation $\rho_0 = j \oplus \mathbf{1}$ is then given by

$$\mathcal{E} = \mathcal{K}_X^{1/2} \oplus \mathcal{O}_X \oplus \mathcal{K}_X^{-1/2}, \quad \Phi_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

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Bronstein–Davalo consider its deformation induced by $q \in \mathrm{H}^0(X, \mathcal{K}_X^{3/2})$,

$$\Phi_q = \begin{pmatrix} 0 & q & 0 \\ 0 & 0 & q \\ 1 & 0 & 0 \end{pmatrix}.$$

Barbot representation and its deformation

The Barbot representation $\rho_0 = j \oplus \mathbf{1}$ is then given by

$$\mathcal{E} = \mathcal{K}_X^{1/2} \oplus \mathcal{O}_X \oplus \mathcal{K}_X^{-1/2}, \quad \Phi_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Bronstein–Davalo consider its deformation induced by $q \in \mathrm{H}^0(X, \mathcal{K}_X^{3/2})$,

$$\Phi_q = \begin{pmatrix} 0 & q & 0 \\ 0 & 0 & q \\ 1 & 0 & 0 \end{pmatrix}.$$

Theorem (Bronstein–Davalo, 2026)

$(\mathcal{E}, \Phi_q)$ is stable when $q \neq 0$. Through the non-Abelian Hodge correspondence,

$$\mathrm{NAH}(\mathcal{E}, \Phi_q) \colon \pi_1(S) \to \mathrm{SL}_3\mathbb{R}$$

is Borel-Anosov for any q.

Fix a $q \in \mathrm{H}^0(X, \mathcal{K}_X^{3/2})$. We write ρ_t for $\mathrm{NAH}(\mathcal{E}, \Phi_{tq})$, where $t \in \mathbb{R}$. We have the following domination proposition.

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Proposition (Yao–Z., 2026+)

There exists $\varepsilon_0 = \varepsilon_0(q) > 0$ *and* $m_0 = m_0(q) > 0$ *such that for any* $t \in (-\varepsilon_0, \varepsilon_0)$ *and* $\gamma \in \pi_1(S)$,

$$\log|\lambda_1(\rho_t(\gamma))| \geqslant \left(\frac{1}{2} + m_0 t^2\right) \cdot \ell_X(\gamma), \quad -\log|\lambda_3(\rho_t(\gamma))| \geqslant \left(\frac{1}{2} + m_0 t^2\right) \cdot \ell_X(\gamma).$$

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As a corollary, we obtain that

Theorem (Yao–Z., 2026+)

Let $j_X : \pi_1(S) \to \mathrm{SL}_2\mathbb{R}$ *be the Fuchsian representation corresponding to* X. *Then there exists* $\varepsilon_1 = \varepsilon_1(q) > 0$ *such that for any* $t \in \mathbb{R}, |t| \in (0, \varepsilon_1)$,

$$(\rho_t \otimes (\tau_k \circ j_X)) : \pi_1(S) \to \mathrm{SL}_{3k}\mathbb{R}$$

is irreducible, Borel-Anosov but non-Hitchin.

Sketch of Proof: Everything varies real-analytically. We study the variation of $a_t(\gamma) := \log|\lambda_1(\rho_t(\gamma))|$.

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- The harmonic metric on $\mathcal{K}_X^{1/2} \oplus \mathcal{O}_X \oplus \mathcal{K}_X^{-1/2}$ is of the form $(\mathrm{e}^{-u_t} h_0, 1, \mathrm{e}^{u_t} h_0^{-1})$.

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- Hitchin's self-duality equation reduced to

$$\Delta_{g_0} u_t = 4(\mathrm{e}^{2u_t} - 1 - t^2 \mathrm{e}^{-u_t} \cdot |q|_{g_0}^2),$$

where g_0 is the conformal hyperbolic metric.

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- Higher order variations can be controlled uniformly by analyticity using the thermodynamic formalism.

Hausdorff dimension of the limit set

The limit set $\Lambda^1_\rho \subset \mathbb{RP}^2$ of a Borel-Anosov representation $\rho\colon \pi_1(S) \to \mathrm{SL}_3\mathbb{R}$ is defined as the closure of the attractive eigenlines of $\rho(\gamma)$ for all $\gamma \in \pi_1(S) \setminus \{1\}$.

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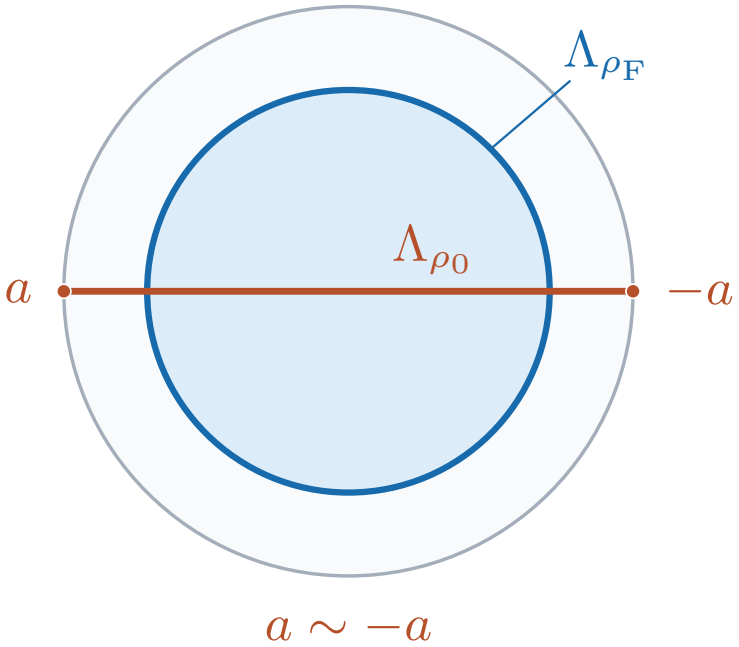

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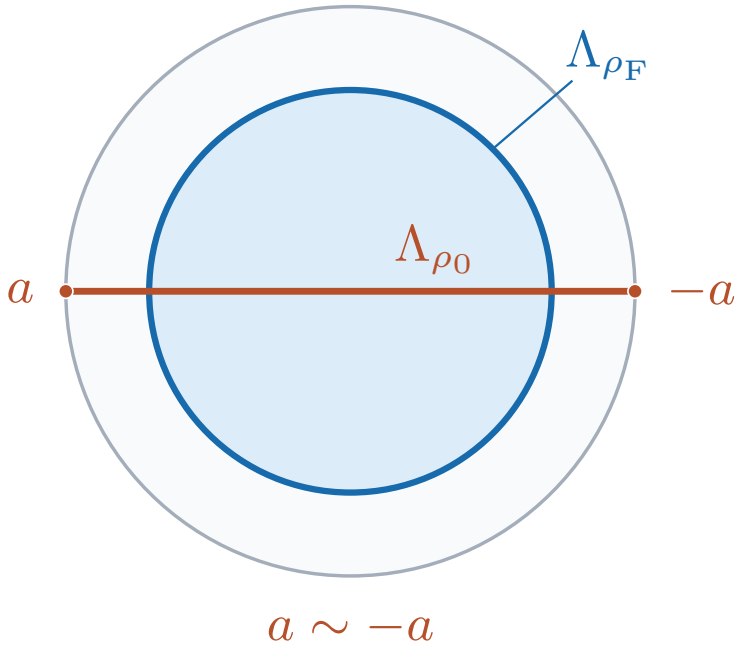

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Both limit sets have Hausdorff dimension 1.
Moreover, the limit set of a Hitchin representation is a C^1-curve (Labourie, 2006).
Its Hausdorff dimension is 1.

Theorem (Li–Pan–Xu, 2023+)

For every $\varepsilon > 0$, there exists a small neighborhood O of ρ_0 such that for any ρ in O, we have

- *either $\rho\colon \pi_1(S) \to \mathrm{SL}_3\mathbb{R}$ is not irreducible;*
- *or ρ is irreducible and*

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Theorem (Yao–Z., 2026+)

There exist $\varepsilon_2 = \varepsilon_2(q) > 0$, $m_1 = m_1(q) > 0$ and $m_1' = m_1'(q) > 0$ such that for any $t \in \mathbb{R}$ with $|t| \in (0, \varepsilon_2)$,

$$\frac{3}{2} - m_1' t^2 \leqslant \dim_{\mathrm{H}}(\Lambda^1_{\rho_t}) \leqslant \frac{3}{2} - m_1 t^2.$$

Moreover, for every small neighborhood O of ρ_0, $3/2$ is an interior point of the set

$$\{\dim_{\mathrm{H}}(\Lambda^1_\rho) \mid \rho \in O\}.$$

Thank you!