4-cyclic Higgs bundles, ∂-alternating surfaces and QIE representations

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Higher (rank) Teichmüller theory

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Moreover, $\mathcal{T}(S)$ is one of the two connected components which consist entirely of discrete and faithful representations. The other one is $\mathcal{T}(\overline{S})$, where $\overline{S}$ denotes the surface S with the opposite orientation.
NO other component contains a discrete and faithful representation.

Orbit map is a quasi-isometric embedding

A representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{R}$ induces an orbit map

$$\begin{aligned} \tau_\rho\colon \pi_1(S) &\longrightarrow \mathrm{PSL}_2\mathbb{R}/\mathrm{PSO}_2\mathbb{R} \cong \mathbb{H}^2 \\ \gamma &\longmapsto \rho(\gamma)\cdot x_0 \end{aligned}$$

for the fixed base point $x_0 = (0,1) \in \mathbb{H}^2$.

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When ρ is Fuchsian, the orbit map is a **quasi-isometric embedding** (QIE), i.e.

$$d_{\mathbb{H}^2}(\tau_\rho(\gamma_1), \tau_\rho(\gamma_2))$$

is bounded by the word distance between γ_1 and γ_2 linearly from above and below.

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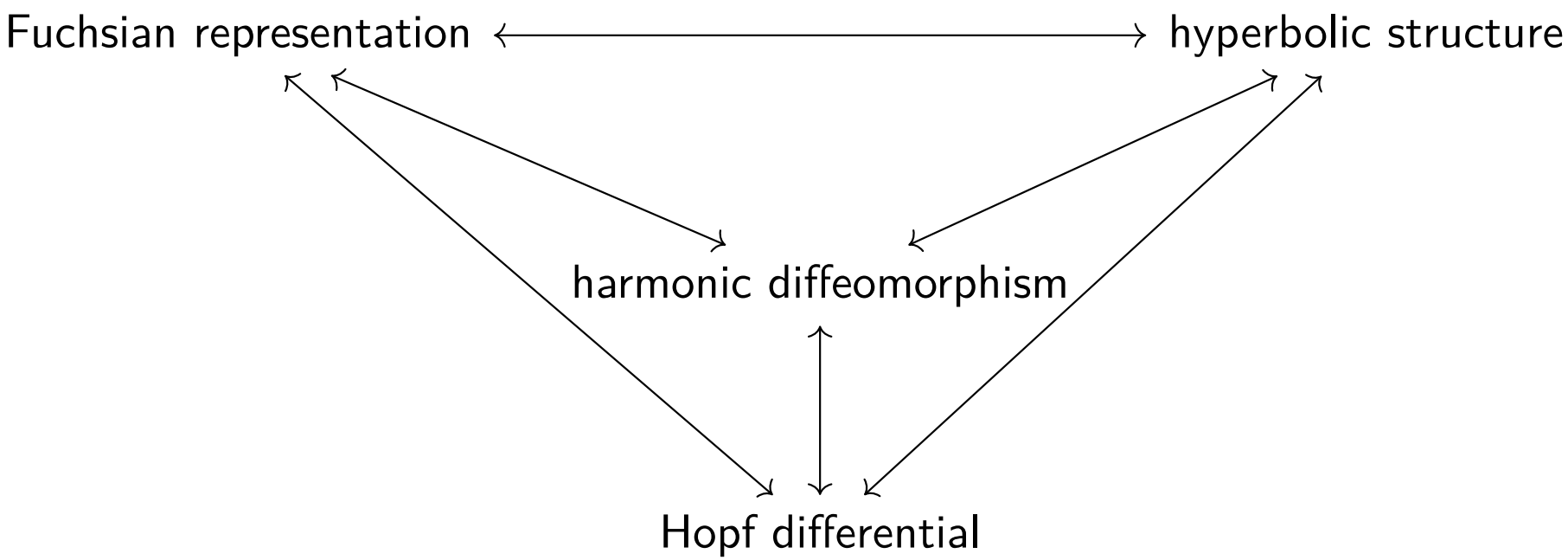
- q is the Hopf differential of f_q.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hyperbolic structure
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In general, the set of discrete and faithful representations is only closed and NOT open.

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- (Hitchin, 1992; Labourie, 2006; Fock–Goncharov, 2006) Hitchin components $\mathcal{T}_{Hit}(S, G)$ for **real split** G, such as $\mathrm{SL}_n\mathbb{R}$, $\mathrm{SO}^0_{n,n+1}$, $\mathrm{Sp}_{2n}\mathbb{R}$ and so on;

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- (Burger–Iozzi–Wienhard, 2010) Maximal components $\mathcal{T}_{max}(S, G)$ for **Hermitian** G, such as $\mathrm{SU}_{p,q}$, $\mathrm{SO}^0_{2,n}$ and so on.

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When $G = \mathrm{PSL}_2\mathbb{R}$, the above components coincide with the classical Teichmüller space.

Toledo invariant

When G is of Hermitian type, i.e. the Riemannian symmetric space G/K of G admits a G-invariant (normalized) Kähler form ω_G, there is a locally constant function, called **Toledo invariant**, on the character variety

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- $\mathrm{Tol}(\rho) \in \frac{1}{q_G}\mathbb{Z}$ *for some integer* q_G.
- ***(Milnor–Wood inequality)***

$$|\mathrm{Tol}(\rho)| \leqslant \mathrm{rank}_{\mathbb{R}}(G/K)\cdot(2g-2).$$

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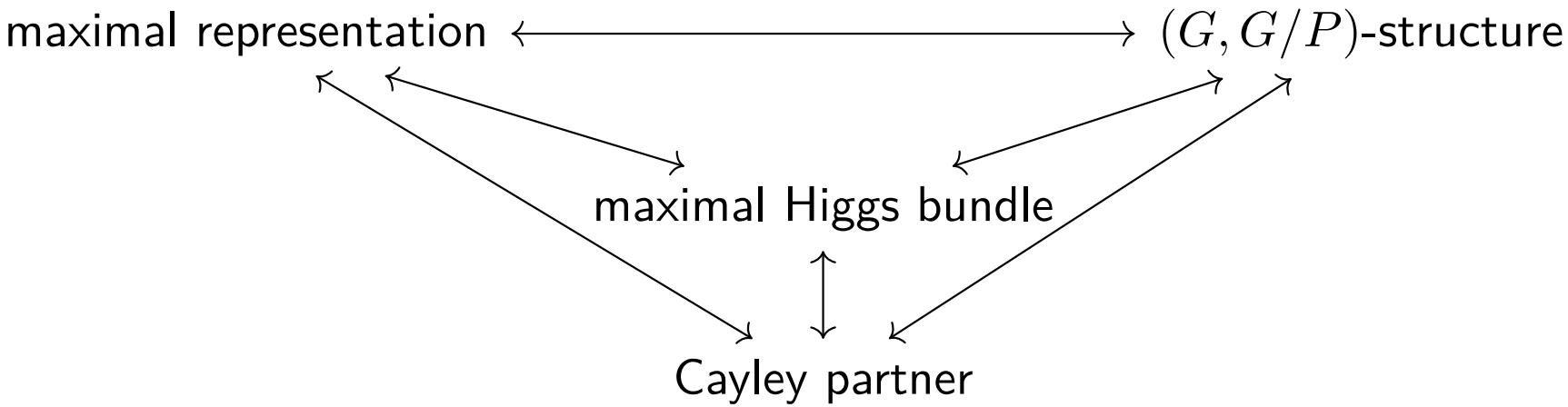
We have a similar picture of maximal components as the Teichmüller space by

- Burger–Iozzi–Wienhard, 2010;
- Guichard–Wienhard, 2012;
- Biquard–García-Prada–Rubio, 2017;
- Alessandrini–Maloni–Tholozan–Wienhard, 2025;
- many others...



Here G/P is the Shilov boundary, and the corresponding structure is defined on a fibre bundle over S with compact fibre.

Good slice in non-maximal components

When $G = \mathrm{PU}_{2,n+1}$, we obtain that

$$\mathrm{Tol}(\rho) \in \frac{2\gcd(n+1,2)}{n+3}\mathbb{Z} \cap \big[-4(g-1),\, 4(g-1)\big] =: \mathscr{T}$$

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Theorem (Li–Z., 2026+)

*For every $t \in \mathscr{T}$ with $|t| \neq 4(g-1)$, the subvariety $\mathfrak{X}_t$ of $\mathfrak{X}(S, \mathrm{PU}_{2,n+1})$ with Toledo invariant t contains a slice $\mathscr{S}_t$ consisting of **irreducible** representations with **quasi-isometric embedded** orbit maps. Its real codimension in the smooth locus of $\mathfrak{X}_t$ is*

$$\mathrm{codim}_{\mathbb{R}}\, \mathscr{S}_t = \begin{cases} 10(g-1), & (n+3)t \in 4\mathbb{Z}, \\ 10(g-1) + 2(n-1), & \text{otherwise.} \end{cases}$$

In particular, every representation in $\mathscr{S}_t$ is discrete and faithful.

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Such slice is obtained by ∂-alternating surfaces. We also have a $(G, G/P)$-structure interpretation.

Higgs bundle and maximal spacelike immersion in $\mathbb{H}^{2,q}$

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A harmonic metric induces a flat connection

$$\mathrm{D}^h = \nabla^h + \Phi + \Phi^{*_h}$$

so gives a representation $\rho := \mathrm{Hol}(\mathrm{D}^h)$. The harmonic metric is further equivalent to a ρ-equivariant harmonic map

$$h \colon \widetilde{X} \to \mathrm{GL}_n\mathbb{C}/\mathrm{U}_n.$$

Pseudo-hyperbolic space

Let

$$\mathbb{R}^{2,q+1} := (\mathbb{R}^{2+(q+1)}, \mathbf{b}),$$

where

$$\mathbf{b}(v,w) = v_1w_1 + v_2w_2 - \sum_{j=3}^{q+3} v_jw_j$$

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The bilinear form $\mathbf{b}$ induces the pseudo-Riemannian metric $g_{\mathbb{H}^{2,q}}$ on $\mathbb{H}^{2,q}$ with signature $(2,q)$.

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A spacelike immersion f defines the following decomposition

$$\begin{array}{cccccccc} \mathbb{R}^{2,q+1} & = & U_p & \oplus & \ell_p & \oplus & N_p \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \text{tangent 2-plane} & & \mathbb{R}\cdot f(p) & & \text{normal space} \\ \text{sign} & & + & & - & & - \end{array}$$

at every $p \in X$.

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- $\mathrm{II}\colon TX \to \mathrm{Hom}(U, N)$ is the **second fundamental form**;
- X is a Riemann surface $\Longrightarrow$ $(\nabla^U)^{0,1}, (\nabla^\ell)^{0,1}, (\nabla^N)^{0,1}$ defines holomorphic structures on $U \otimes \mathbb{C}$, $\ell \otimes \mathbb{C}$, $N \otimes \mathbb{C}$ respectively. We denote the resulting holomorphic bundles by $\mathcal{U}$, $\mathcal{L}$, $\mathcal{N}$ respectively.

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where f is a ρ-equivariant maximal spacelike immersion, then we have the corresponding 4-cyclic Higgs bundle:

$$(\mathcal{E}_f, \Phi_f) := \begin{array}{ccccc} \mathcal{U}^{0,1} & \xrightarrow{\bar\partial f} & \mathcal{L} & \xrightarrow{\partial f} & \mathcal{U}^{1,0} \\ & \mathrm{II}^{0,2} \nwarrow & & \nearrow \mathrm{II}^{2,0} & \\ & & \mathcal{N} & & \end{array}$$

with a natural ρ-equivariant harmonic map $h\colon \widetilde{X} \to \mathrm{Sym}(\mathrm{SO}^0_{2,q+1})$ induced by a generalized Gauss map, and $\rho = \mathrm{Hol}(\mathrm{D}^h)$.

Anosov property and domain of discontinuity

Theorem (Collier–Tholozan–Toulisse, 2019)

Given

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- *The total space of the fibre bundle*

$$\mathrm{Pho}(U \oplus N) \to X$$

is diffeomorphic to

$$\rho(\pi_1(X))\backslash\Omega_\rho$$

for an open domain $\Omega_\rho \subset \mathrm{Pho}(\mathbb{R}^{2,q+1})$, where Pho denotes the space of isotropic 2-planes (photons).

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Definition

$\rho\colon \pi_1(X) \to \mathrm{GL}_n\mathbb{C}$ is P_k-**Anosov** if there exist constants $D, L > 0$ such that

$$\log \frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))} \geqslant D \cdot d_{\mathbb{H}^2}(x_0, \gamma \cdot x_0) - L, \forall \gamma \in \pi_1(X),$$

where σ_i denotes the i-th singular value.

∂-alternating surfaces in $\mathbb{CH}^{m,n}$

Complex pseudo-hyperbolic space

Let

$$\mathbb{C}^{m,n+1} := (\mathbb{C}^{m+(n+1)}, H),$$

where

$$H(v,w) = \sum_{j=1}^{m} \overline{v_j} w_j - \sum_{j=m+1}^{m+n+1} \overline{v_j} w_j$$

is an Hermitian form with signature $(m, n+1)$.

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The Hermitian form H induces the pseudo-Kähler metric H_0 on $\mathbb{CH}^{m,n}$ with signature (m,n).

∂-alternating surfaces

Let $f: X \to \mathbb{CH}^{m,n}$ be a smooth immersion from a Riemann surface X (maybe not compact).

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$$\partial\text{-alternating}$$
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$$\text{conformal} \quad + \quad \text{spacelike in } (1,0)\text{-direction} \quad + \quad \text{harmonic}$$

Similar to maximal spacelike immersion in $\mathbb{H}^{2,q}$, by decomposing the pullback bundle of $T^{1,0}\mathbb{CH}^{m,n}$ we obtain a harmonic 4-cyclic Higgs bundle of the following form over X.

$$(\mathcal{E}_f, \Phi_f) := \begin{array}{ccccc} \mathcal{N}_+ & \xrightarrow{(\bar\partial^{1,0} f)^* H_0} & \mathcal{L} & \xrightarrow{\partial^{1,0} f} & \mathcal{T}_+ \\ & \nwarrow^{\beta} & & \swarrow_{\mathrm{II}^{2,0}} & \\ & & \mathcal{N}_- & & \end{array}$$

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Proposition (Li–Z., 2026+)

Every harmonic 4-cyclic Higgs bundle of the form above defines a ∂-alternating surface on X.

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- $\mathrm{II}^{2,0}, \beta \equiv 0 \implies f$ is a minimal immersion in $\mathbb{CH}^{m,0}$;
- $f\colon X \to \mathbb{H}^{2,q}$ spacelike conformal immersion, then

$$f \text{ is } \partial\text{-alternating in } \mathbb{CH}^{2,q} \iff f \text{ is maximal.}$$

Equivariant case

Let X be a closed Riemann surface whose genus $\geqslant 2$.

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Theorem (Li–Z., 2026+)

The generalized Gauss map

$$\begin{aligned} \mathcal{G}_f\colon \widetilde{X} &\longrightarrow \{\textit{positive } m\textit{-planes in } \mathbb{C}^{m,n+1}\} = \mathrm{Sym}(\mathrm{PU}_{m,n+1}) \\ x &\longmapsto (\mathcal{T}_{+,x} \oplus \mathcal{N}_{+,x}) \end{aligned}$$

is the associated harmonic map of $(\mathcal{E}_f, \Phi_f)$.

Bi-Lipschitz embedding & domain of discontinuity

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Theorem (Li–Z., 2026+)

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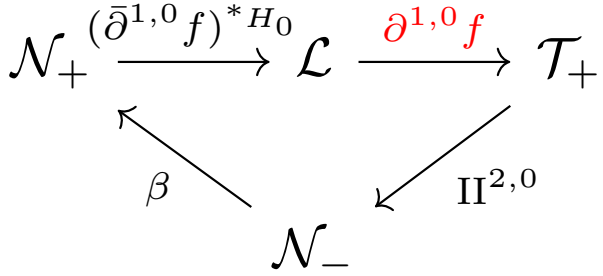
Furthermore, the fibre is diffeomorphic to

$$S^{2n-1} \times S^{2n-1}.$$

For $\mathbb{CH}^{2,n}$ or $\mathbb{CH}^{n+1,1}$, either $\mathcal{N}_+$ or $\mathcal{N}_-$ in



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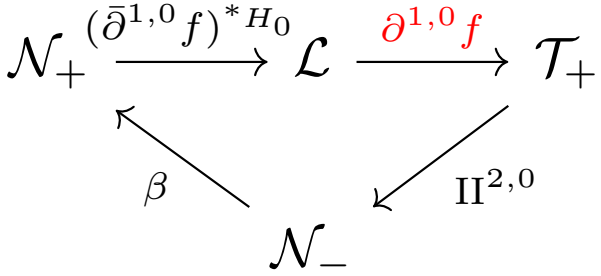

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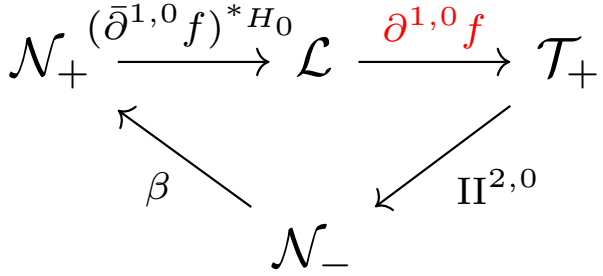

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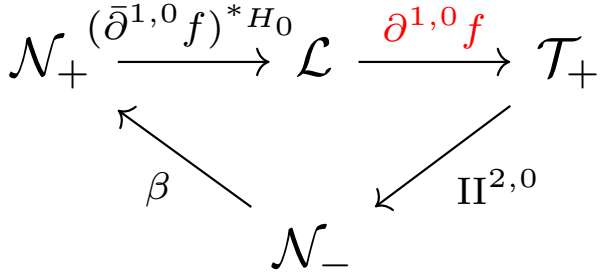

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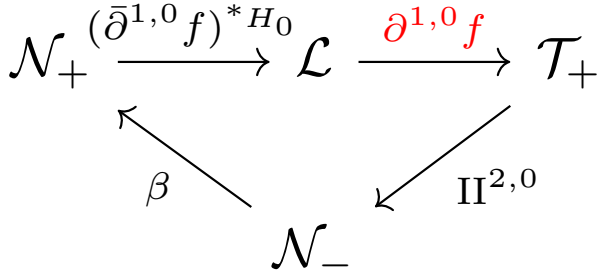

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Thank you!

Sketch of Proof

The core idea is to estimate the norm of the flat section with respect to the associated Higgs bundle $(\wedge^2\mathcal{E}, \wedge^2\Phi)$.

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It suffices to show that there exist constants $C_1, C_2, \varepsilon > 0$ such that

$$\|v(x)\|^2 \geqslant C_1 \cdot \exp(\varepsilon \cdot d_{\mathbb{H}^2}(x, x_0)) - C_2,$$

then the orbit map of ρ is a quasi-isometric embedding.

Filip's estimate (simplified)

Lemma (Filip, 2021+)

Given a complete Riemannian manifold (M, g) with the distance function $d\colon M \times M \to \mathbb{R}$ and a smooth function $f \in C^\infty(M;\mathbb{R})$. Suppose that f satisfies

(S1) *f is a non-negative function with all zeroes isolated;*

(S2) $\|\mathrm{d}f\|_g \gtrsim f$;

(S3) $\|\mathrm{d}f\|_g \gtrsim f^{1/2}$.

Then f achieves its unique minimum at $x_{\min}$ and there exist constants $C_1, C_2, \varepsilon > 0$, such that

$$f(x) \geqslant C_1 \cdot \exp(\varepsilon \cdot d(x_{\min}, x)) - C_2, \forall x \in M.$$

Recall that it suffices to deal with the polystable 4-cyclic Higgs bundle

$$(\mathcal{E},\Phi)= \quad \mathcal{E}_0 \xrightarrow{\mathbf{1}} \mathcal{E}_1 \xrightarrow{\alpha} \mathcal{E}_2 \xrightarrow{\beta} \mathcal{E}_3 \xrightarrow{\gamma} \mathcal{E}_0$$

whose ranks $(\mathrm{rank}(\mathcal{E}_j))=(1,1,q,1)$ and signatures $(-,+,-,+)$.

Part of $(\wedge^2\mathcal{E}, \wedge^2\Phi)$ is

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