4-cyclic Higgs bundles, ∂-alternating surfaces and QIE representations

(in progress, joint with 李琼玲 Qiongling Li)

张峻铭 Junming Zhang

Chern Institute of Mathematics, Nankai University

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南開大學
Nankai University

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Goal: Study the monodromy via harmonic map.

Contents

1. Maximal spacelike immersion in $\mathbb{H}^{2,q}$
2. ∂-alternating surfaces in $\mathbb{CH}^{m,n}$
3. Bi-Lipschitz embedding & domain of discontinuity

Maximal spacelike immersion in $\mathbb{H}^{2,q}$

Pseudo-hyperbolic space

Let

$$\mathbb{R}^{2,q+1} := (\mathbb{R}^{2+(q+1)}, \mathbf{b}),$$

where

$$\mathbf{b}(v,w) = v_1w_1 + v_2w_2 - \sum_{j=3}^{q+3} v_jw_j$$

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The bilinear form $\mathbf{b}$ induces the pseudo-Riemannian metric $g_{\mathbb{H}^{2,q}}$ on $\mathbb{H}^{2,q}$ with signature $(2,q)$.

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A spacelike immersion f defines the following decomposition

$$\begin{array}{ccccccc} \mathbb{R}^{2,q+1} & = & U_p & \oplus & \ell_p & \oplus & N_p \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \text{tangent 2-plane} & & \mathbb{R}\cdot f(p) & & \text{normal space} \\ \text{sign} & & + & & - & & - \end{array}$$

at every $p \in X$.

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- $\mathbf{b}$ extends holomorphically on $\mathcal{U}$, $\mathcal{L}$, $\mathcal{N}$. In particular, $\mathcal{L} \cong \mathcal{L}^{-1}$, $\mathcal{U} \cong \mathcal{U}^\vee$, $\mathcal{N} \cong \mathcal{N}^\vee$.

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the trivial connection

$$\mathrm{D} = \nabla^H + \underbrace{\begin{pmatrix} & \partial f & \\ & & \mathrm{II}^{0,2} \\ \mathrm{II}^{2,0} & & \end{pmatrix}}_{\Phi} - \Phi^{*_H},$$

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$$\begin{cases} f \text{ is harmonic \& Gauß–Codazzi equations} \implies (\nabla^h)^{0,1}\Phi = 0, \\ F(\mathrm{D}) = 0 \implies F(\nabla^h) + [\Phi \wedge \Phi^{*_h}] = 0. \end{cases}$$

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An Hermitian metric h on $\mathcal{E}$ is called **harmonic** if

$$F(\nabla^h)+[\Phi,\Phi^{*_h}]=0. \quad \text{(Hitchin's self-duality equation)}$$

Therefore, a harmonic metric induces a flat connection

$$\mathrm{D}^h=\nabla^h+\Phi+\Phi^{*_h}.$$

In other words, h is the harmonic metric of the following 4-cyclic Higgs bundle.

$$(\mathcal{E}_f, \Phi_f) := \begin{array}{ccccc} \mathcal{L}\mathcal{K}_X & \xrightarrow{\bar{\partial} f} & (\mathcal{L}, \mathbf{b}|_{\mathcal{L}}) & \xrightarrow{\partial f} & \mathcal{L}\mathcal{K}_X^{-1} \\ & \nwarrow^{\mathrm{II}^{0,2}} & & \swarrow_{\mathrm{II}^{2,0}} & \\ & & (\mathcal{N}, \mathbf{b}|_{\mathcal{N}}) & & \end{array}$$

Equivariant case

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$$\left(\rho\colon \pi_1(X) \to \mathrm{SO}^0_{2,q+1}, \quad f\colon \widetilde{X} \to \mathbb{H}^{2,q}\right)$$

where f is a ρ-equivariant maximal spacelike immersion, then $(\mathcal{E}_f, \Phi_f)$ descends to X and

$$\rho = \mathrm{Hol}(\mathrm{D}^h).$$

Anosov property and domain of discontinuity

Theorem (Collier–Tholozan–Toulisse, 2019)

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- *The total space of the fibre bundle*

$$\mathrm{Pho}(U \oplus N) \to X$$

is diffeomorphic to

$$\rho(\pi_1(X))\backslash\Omega_\rho$$

for an open domain $\Omega_\rho \subset \mathrm{Pho}(\mathbb{R}^{2,q+1})$, where Pho denotes the space of isotropic 2-planes (photons).

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$\rho\colon \pi_1(X) \to \mathrm{GL}_n\mathbb{C}$ is P_k-**Anosov** if there exist constants $D, L > 0$ such that

$$\log \frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))} \geqslant D \cdot d_{\mathbb{H}^2}(x_0, \gamma \cdot x_0) - L, \forall \gamma \in \pi_1(X),$$

where σ_i denotes the i-th singular value.

∂-alternating surfaces in $\mathbb{CH}^{m,n}$

Complex pseudo-hyperbolic space

Let

$$\mathbb{C}^{m,n+1} := (\mathbb{C}^{m+(n+1)}, H),$$

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$$H(v,w) = \sum_{j=1}^{m} \overline{v_j} w_j - \sum_{j=m+1}^{m+n+1} \overline{v_j} w_j$$

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The Hermitian form H induces the pseudo-Kähler metric H_0 on $\mathbb{CH}^{m,n}$ with signature (m,n).

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∂-alternating $\Longrightarrow$ spacelike in $(1,0)$-direction.

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$$\partial^{1,0}f \in \Omega^{1,0}(V_f) \quad \text{and} \quad \bar{\partial}^{1,0}f \in \Omega^{0,1}(V_f).$$

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- f is conformal $\implies$ $q_2 := \delta_f(\partial^{1,0}f)$ vanishes;
- f is harmonic and conformal $\implies$ $\sigma^{2,0}$ is the usual $(2,0)$-part of the second fundamental form.

∂-alternating surfaces

A ∂-alternating surface $f\colon X \to \mathbb{CH}^{m,n}$ is a specific conformal and harmonic immersion with positive $T^+ = \langle \partial^{1,0} f \rangle$.

Proposition (Li–Z., 2026+)

Let $f\colon X \to \mathbb{CH}^{m,n}$ be a ∂-alternating surface. Then the trivial bundle $\underline{\mathbb{C}^{m,n+1}}$ over X admits a smooth H-orthorgonal decomposition

$$\underline{\mathbb{C}^{m,n+1}} = L_f \oplus T_+ \oplus N_- \oplus N_+,$$

where N_- is negative and N_+ is positive, such that the trivial connection

$$\mathrm{D} = \nabla^H + \underbrace{\begin{pmatrix} & & \gamma \\ \mathbf{1} & & \\ & \alpha & \\ & & \beta \end{pmatrix}}_{\Phi} - \Phi^{*_H}$$

and $(\nabla^H)^{0,1}\Phi = 0$. Here

$$\mathbf{1} = \partial^{1,0} f, \quad \alpha = \sigma^{2,0}, \quad \beta = ((\nabla^{V_f})^{1,0}|_{N_-})^{N^+} \text{ and } \gamma = \delta_f.$$

Again, by changing the signs of H on L_f and N_-, we obtain a harmonic metric h for the 4-cyclic Higgs bundle of the following form over X.

$$(\mathcal{E}_f, \Phi_f) = \mathcal{L} \xrightarrow{\mathbf{1}} \mathcal{L}\mathcal{K}_X^{-1} \xrightarrow{\alpha} \mathcal{V} \xrightarrow{\beta} \mathcal{W} \xrightarrow{\gamma} \mathcal{L}$$

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Proposition (Li–Z., 2026+)

Every harmonic 4-cyclic Higgs bundle of the form above defines a ∂-alternating surface on X.

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- $f\colon X \to \mathbb{H}^{2,q}$ spacelike conformal immersion, then

$$f \text{ is } \partial\text{-alternating in } \mathbb{CH}^{2,q} \iff f \text{ is maximal.}$$

Equivariant case

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Theorem (Li–Z., 2026+)

The generalized Gauss map

$$\begin{aligned} \mathcal{G}_f\colon \widetilde{X} &\longrightarrow \{\textit{positive } m\textit{-planes in } \mathbb{C}^{m,n+1}\} = \mathrm{Sym}(\mathrm{PU}_{m,n+1}) \\ x &\longmapsto (T_{+,x} \oplus N_{+,x}) \end{aligned}$$

is the associated harmonic map of $(\mathcal{E}_f, \Phi_f)$.

Bi-Lipschitz embedding & domain of discontinuity

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- *The orbit map*

$$\begin{aligned} \tau_\rho \colon \pi_1(X) &\longrightarrow \mathrm{Sym}(\mathrm{PU}_{2,n+1}) \\ \gamma &\longmapsto \rho(\gamma) \cdot x_0 \end{aligned}$$

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Toledo invariant

When G is of Hermitian type, i.e. the Riemannian symmetric space G/K of G admits a G-invariant (normalized) Kähler form ω_G, the connected components of the character variety

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$$|\mathrm{Tol}(\rho)| \leqslant \mathrm{rank}_{\mathbb{R}}(G/K)\cdot(2g-2).$$

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For every $t \in \mathscr{T}$ with $|t| \neq 4(g-1)$, the subvariety $\mathfrak{X}_t$ of $\mathfrak{X}(X, \mathrm{PU}_{2,n+1})$ with Toledo invariant t contains a slice $\mathscr{S}_t$ consisting of irreducible representations with quasi-isometric embedded orbit maps. Its real codimension in the smooth locus of $\mathfrak{X}_t$ is

$$\operatorname{codim}_{\mathbb{R}} \mathscr{S}_t = \begin{cases} 10(g-1), & (n+3)t \in 4\mathbb{Z}, \\ 10(g-1)+2(n-1), & \text{otherwise.} \end{cases}$$

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Remark

The slice is obtained by ∂-alternating surfaces + varying complex structure of X.

Domain of discontinuity

Theorem (Li–Z., 2026+)

Let (ρ, f) be an equivariant ∂-alternating surface to $\mathbb{CH}^{2,n}$ or $\mathbb{CH}^{n+1,1}$ on X and ρ is non-maximal.

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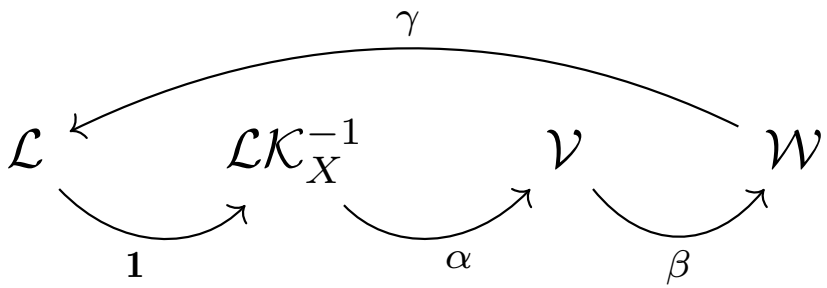
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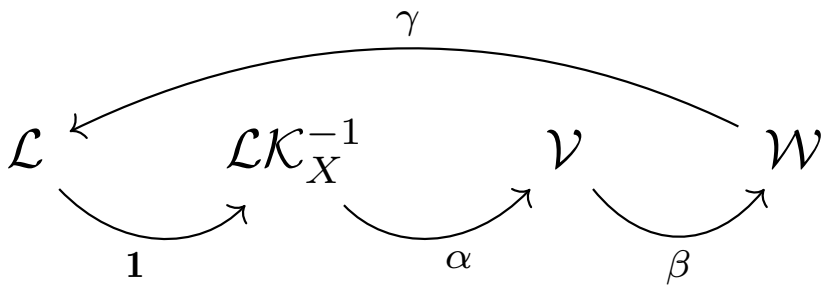
$$S^{2n-1} \times S^{2n-1}.$$

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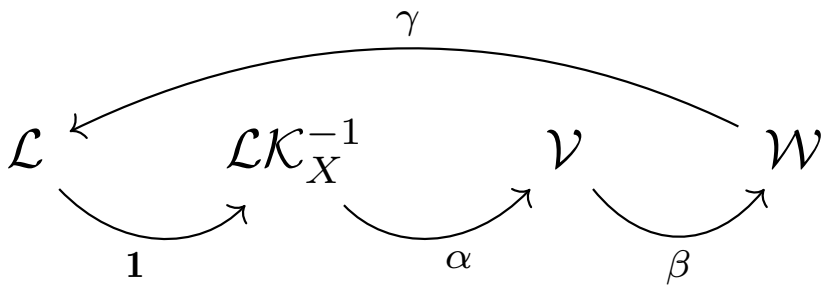
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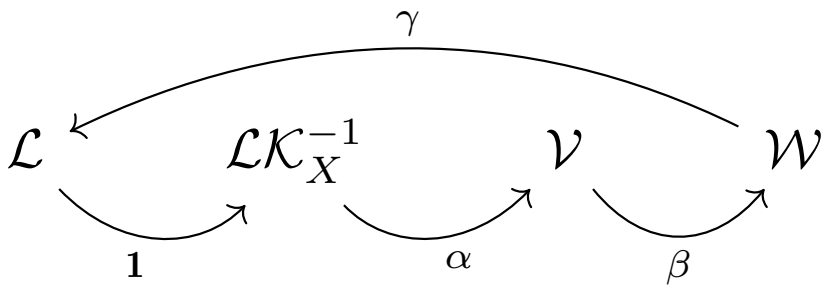


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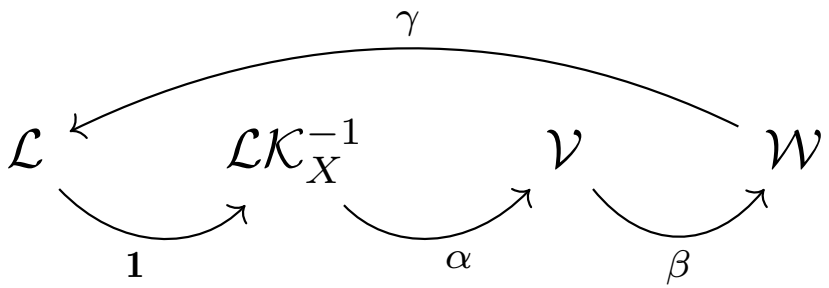
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When $n = 1$,

- *(Filip, 2021+) $\beta = 0 \implies$ ρ is P_1-Anosov;*

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Sketch of Proof

The core idea is to estimate the norm of the flat section with respect to the associated Higgs bundle $(\wedge^2\mathcal{E}, \wedge^2\Phi)$.

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$$\text{estimate the sum of singular values of } \rho(\gamma) \rightsquigarrow \|(\wedge^2\rho)(\gamma)\cdot v\|^2 \downarrow \|v(\gamma\cdot x_0)\|^2$$

It suffices to show that there exist constants $C_1, C_2, \varepsilon > 0$ such that

$$\|v(x)\|^2 \geqslant C_1 \cdot \exp(\varepsilon \cdot d_{\mathbb{H}^2}(x, x_0)) - C_2,$$

then the orbit map of ρ is a quasi-isometric embedding.

Filip's estimate (simplified)

Lemma (Filip, 2021+)

Given a complete Riemannian manifold (M, g) with the distance function $d\colon M \times M \to \mathbb{R}$ and a smooth function $f \in C^\infty(M;\mathbb{R})$. Suppose that f satisfies

(S1) *f is a non-negative function with all zeroes isolated;*

(S2) $\|\mathrm{d}f\|_g \gtrsim f$;

(S3) $\|\mathrm{d}f\|_g \gtrsim f^{1/2}$.

Then f achieves its unique minimum at $x_{\min}$ and there exist constants $C_1, C_2, \varepsilon > 0$, such that

$$f(x) \geqslant C_1 \cdot \exp(\varepsilon \cdot d(x_{\min}, x)) - C_2, \forall x \in M.$$

Recall that it suffices to deal with the polystable 4-cyclic Higgs bundle

$$(\mathcal{E},\Phi)= \quad \mathcal{E}_0 \xrightarrow{\mathbf{1}} \mathcal{E}_1 \xrightarrow{\alpha} \mathcal{E}_2 \xrightarrow{\beta} \mathcal{E}_3 \xrightarrow{\gamma} \mathcal{E}_0$$

whose ranks $(\mathrm{rank}(\mathcal{E}_j))=(1,1,q,1)$ and signatures $(-,+,-,+)$.

Part of $(\wedge^2\mathcal{E}, \wedge^2\Phi)$ is

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Thank you!