# 4-cyclic Higgs bundles, $\partial$-alternating surfaces and QIE representations

(in progress, joint with 李琼玲 Qiongling Li)

张峻铭 Junming Zhang

Chern Institute of Mathematics, Nankai University

Workshop on Classical and Higher Teichmüller theory
at Nankai University, Tianjin




南開大學
Nankai University

harmonic map $\Longleftrightarrow$ Higgs bundle

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**Goal: Study the monodromy via harmonic map.**

# Contents


1. Maximal spacelike immersion in $\mathbb{H}^{2,q}$
2. $\partial$-alternating surfaces in $\mathbb{CH}^{m,n}$
3. Bi-Lipschitz embedding & domain of discontinuity

# Maximal spacelike immersion in $\mathbb{H}^{2,q}$

# Pseudo-hyperbolic space

Let

$$\mathbb{R}^{2,q+1} := (\mathbb{R}^{2+(q+1)}, \mathbf{b}),$$

where

$$\mathbf{b}(v,w) = v_1w_1 + v_2w_2 - \sum_{j=3}^{q+3} v_jw_j$$

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The bilinear form $\mathbf{b}$ induces the pseudo-Riemannian metric $g_{\mathbb{H}^{2,q}}$ on $\mathbb{H}^{2,q}$ with signature $(2,q)$.

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A spacelike immersion $f$ defines the following decomposition

$$\begin{array}{ccccccc} \mathbb{R}^{2,q+1} & = & U_p & \oplus & \ell_p & \oplus & N_p \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \text{tangent 2-plane} & & \mathbb{R}\cdot f(p) & & \text{normal space} \\ \text{sign} & & + & & - & & - \end{array}$$

at every $p \in X$.

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- $\mathbf{b}$ extends holomorphically on $\mathcal{U}$, $\mathcal{L}$, $\mathcal{N}$. In particular, $\mathcal{L} \cong \mathcal{L}^{-1}$, $\mathcal{U} \cong \mathcal{U}^\vee$, $\mathcal{N} \cong \mathcal{N}^\vee$.

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$$\mathrm{d}f = \partial f + \bar{\partial} f,$$

where

$$\partial f\colon T^{1,0}X \to \mathrm{Hom}(\ell \otimes \mathbb{C}, \mathcal{U}^{1,0}) \text{ and } \bar{\partial} f\colon T^{0,1}X \to \mathrm{Hom}(\ell \otimes \mathbb{C}, \mathcal{U}^{0,1})$$

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Therefore, $\mathcal{U}^{1,0} \cong \mathcal{LK}_X^{-1}$ and $\mathcal{U}^{0,1} \cong \mathcal{LK}_X$.

# Holomorphicity of the second fundamental form

The mean curvature of $f$ vanishes implies that

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does not has $(1,1)$-part, where

$$\mathrm{II}^{2,0} : T^{1,0}X \to \mathrm{Hom}(\mathcal{U}^{1,0}, N \otimes \mathbb{C})$$

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Then, $\mathbb{H}^{2,q}$ has constant scalar curvature implies that, by Codazzi equation,

$\mathrm{II}^{2,0}$ is a holomorphic section of $\mathrm{Hom}(\mathcal{K}_X^{-1}, \mathrm{Hom}(\mathcal{U}^{1,0}, \mathcal{N})) \cong \mathcal{L}\mathcal{K}_X^2\mathcal{N}$.

Also, $\mathrm{II}^{0,2}$ is holomorphic.

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Changing the sign of $H$ on $\mathcal{L} \oplus \mathcal{N}$, we obtain an Hermitian metric $h$ on $\underline{\mathbb{R}^{2,q+1}} \otimes \mathbb{C}$. Then

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An Hermitian metric $h$ on $\mathcal{E}$ is called **harmonic** if

$$F(\nabla^h)+[\Phi,\Phi^{*_h}]=0. \quad \text{(Hitchin's self duality equation)} \tag{1}$$

Therefore, a harmonic metric induces a flat connection

$$\mathrm{D}^h=\nabla^h+\Phi+\Phi^{*_h}.$$

In other words, $h$ is the harmonic metric of the following $4$-cyclic Higgs bundle.

$$(\mathcal{E}_f, \Phi_f) := \begin{array}{ccccc} \mathcal{L}\mathcal{K}_X & \xrightarrow{\bar{\partial} f} & (\mathcal{L}, \mathbf{b}|_{\mathcal{L}}) & \xrightarrow{\partial f} & \mathcal{L}\mathcal{K}_X^{-1} \\ & \nwarrow^{\mathrm{II}^{0,2}} & & \swarrow_{\mathrm{II}^{2,0}} & \\ & & (\mathcal{N}, \mathbf{b}|_{\mathcal{N}}) & & \end{array}$$

# Equivariant case

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$$\left(\rho\colon \pi_1(X) \to \mathrm{SO}^0_{2,q+1}, \quad f\colon \widetilde{X} \to \mathbb{H}^{2,q}\right)$$

where $f$ is a $\rho$-equivariant maximal spacelike immersion, then $(\mathcal{E}_f, \Phi_f)$ descends to $X$ and

$$\rho = \mathrm{Hol}(\mathrm{D}^h).$$

# Anosov property and domain of discontinuity

**Theorem (Collier–Tholozan–Toulisse, 2019)**

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- *$\rho$ has maximal Toledo invariant, hence is $P_1$-**Anosov** ($\Longrightarrow$ **discrete** + **faithful**);*
- *The total space of the fibre bundle*

$$\mathrm{Pho}(U \oplus N) \to X$$

*is diffeomorphic to*

$$\rho(\pi_1(X))\backslash\Omega_\rho$$

*for an open domain $\Omega_\rho \subset \mathrm{Pho}(\mathbb{R}^{2,q+1})$, where $\mathrm{Pho}$ denotes the space of isotropic 2-planes (photons).*

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$\rho\colon \pi_1(X) \to \mathrm{GL}_n\mathbb{C}$ is $P_k$-**Anosov** if there exist constants $D, L > 0$ such that

$$\log \frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))} \geqslant D \cdot d_{\mathbb{H}^2}(x_0, \gamma \cdot x_0) - L, \forall \gamma \in \pi_1(X),$$

where $\sigma_i$ denotes the $i$-th singular value.

# $\partial$-alternating surfaces in $\mathbb{CH}^{m,n}$

# Complex pseudo-hyperbolic space

Let

$$\mathbb{C}^{m,n+1} := (\mathbb{C}^{m+(n+1)}, H),$$

where

$$H(v,w) = \sum_{j=1}^{m} \overline{v_j} w_j - \sum_{j=m+1}^{m+n+1} \overline{v_j} w_j$$

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The Hermitian form $H$ induces the pseudo-Kähler metric $H_0$ on $\mathbb{CH}^{m,n}$ with signature $(m,n)$.

# $\partial$-alternating surfaces

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**$\partial$-alternating $\Longrightarrow$ spacelike in $(1,0)$-direction.**

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We consider the $(1,0)$-part $\mathrm{d}^{1,0}f$ of $\mathrm{d}f$, which sends $TX$ to $V_f$. It further decomposes as

$$\partial^{1,0}f \in \Omega^{1,0}(V_f) \quad \text{and} \quad \bar{\partial}^{1,0}f \in \Omega^{0,1}(V_f).$$

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Define

$$\delta_f(\cdot) := H_0(\bar{\partial}^{1,0}f, \cdot) \in \Omega^{1,0}(V_f^\vee),$$

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$$\partial^{1,0}f \in \Omega^{1,0}(V_f) \quad \text{and} \quad \bar{\partial}^{1,0}f \in \Omega^{0,1}(V_f).$$

Let $T^+$ be the subbundle generated by $\partial^{1,0}f$.

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$$\delta_f(\cdot) := H_0(\bar{\partial}^{1,0}f, \cdot) \in \Omega^{1,0}(V_f^\vee), \quad \sigma^{2,0} := ((\nabla^{V_f})^{1,0})^{\perp_{T_+}}$$

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- $f$ is conformal $\implies$ $q_2 := \delta_f(\partial^{1,0}f)$ vanishes;
- $f$ is harmonic and conformal $\implies$ $\sigma^{2,0}$ is the usual $(2,0)$-part of the second fundamental form.

# $\partial$-alternating surfaces

A $\partial$-alternating surface $f\colon X \to \mathbb{CH}^{m,n}$ is a specific conformal and harmonic immersion with positive $T^+ = \langle \partial^{1,0} f \rangle$.

## Proposition (Li–Z., 2026+)

*Let $f\colon X \to \mathbb{CH}^{m,n}$ be a $\partial$-alternating surface. Then the trivial bundle $\underline{\mathbb{C}^{m,n+1}}$ over $X$ admits a smooth $H$-orthorgonal decomposition*

$$\underline{\mathbb{C}^{m,n+1}} = L_f \oplus T_+ \oplus N_- \oplus N_+,$$

*where $N_-$ is negative and $N_+$ is positive, such that the trivial connection*

$$\mathrm{D} = \nabla^H + \underbrace{\begin{pmatrix} & & \gamma \\ \mathbf{1} & & \\ & \alpha & \\ & & \beta \end{pmatrix}}_{\Phi} - \Phi^{*_H}$$

*and $(\nabla^H)^{0,1}\Phi = 0$. Here*

$$\mathbf{1} = \partial^{1,0} f, \quad \alpha = \sigma^{2,0}, \quad \beta = ((\nabla^{V_f})^{1,0}|_{N_-})^{N^+} \text{ and } \gamma = \delta_f.$$

Again, by changing the signs of $H$ on $L_f$ and $N_-$, we obtain a harmonic metric $h$ for the $4$-cyclic Higgs bundle of the following form over $X$.

$$(\mathcal{E}_f, \Phi_f) = \mathcal{L} \xrightarrow{\mathbf{1}} \mathcal{L}\mathcal{K}_X^{-1} \xrightarrow{\alpha} \mathcal{V} \xrightarrow{\beta} \mathcal{W} \xrightarrow{\gamma} \mathcal{L}$$

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**Proposition (Li–Z., 2026+)**

*Every harmonic $4$-cyclic Higgs bundle of the form above defines a $\partial$-alternating surface on $X$.*

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$$f \text{ is } \partial\text{-alternating in } \mathbb{CH}^{2,q} \iff f \text{ is maximal.}$$

# Equivariant case

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$$\left(\rho\colon \pi_1(X) \to \mathrm{PU}_{m,n+1}, \quad f\colon \widetilde{X} \to \mathbb{CH}^{m,n}\right)$$

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**Theorem (Li–Z., 2026+)**

*The generalized Gauss map*

$$\begin{aligned} \mathcal{G}_f\colon \widetilde{X} &\longrightarrow \{\textit{positive } m\textit{-planes in } \mathbb{C}^{m,n+1}\} = \mathrm{Sym}(\mathrm{PU}_{m,n+1}) \\ x &\longmapsto (T_{+,x} \oplus N_{+,x}) \end{aligned}$$

*is the associated harmonic map of* $(\mathcal{E}_f, \Phi_f)$.

# Bi-Lipschitz embedding & domain of discontinuity

**Theorem (Li–Z., 2026+)**

*Let $(\rho, f)$ be an equivariant $\partial$-alternating surface to $\mathbb{CH}^{2,q}$ or $\mathbb{CH}^{q+1,1}$ on a closed Riemann surface $X$ whose genus $\geqslant 2$. Then*

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# Domain of discontinuity

Let $\mathcal{S}_i(\mathbb{C}^{m,n+1}) \subset \mathrm{Gr}_2(\mathbb{C}^{m,n+1})$ denote the space consisting of positive semi-defnite 2-planes with signature $(i, 0|2-i)$.

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$$\{P_x \in \mathcal{S}_i(\mathbb{C}^{2,q+1}) \mid P_x \cap (T_{+,x} \oplus N_{-,x}) \neq \{0\}\}$$

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*for open domains $\Omega_i \subset \mathcal{S}_i(\mathbb{C}^{2,q+1})$, and moreover $\Omega_2 = \mathcal{S}_2(\mathbb{C}^{2,q+1})$. Furthermore, the diffeomorphic types of the fibre $F_i$ of $\mathcal{F}_i$ are*

$$F_0 \cong S^{2q-1} \times S^{2q-1}, \quad F_1 \cong S^{4q+1} \setminus (S^{2q-1} \times S^{2q-1}), \quad F_2 \cong \mathbb{R}^{4q+2}.$$

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$$\beta \text{ is nowhere vanishing} \implies \rho \text{ is maximal} \implies \rho \text{ is } P_2\text{-Anosov.}$$

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# Sketch of Proof

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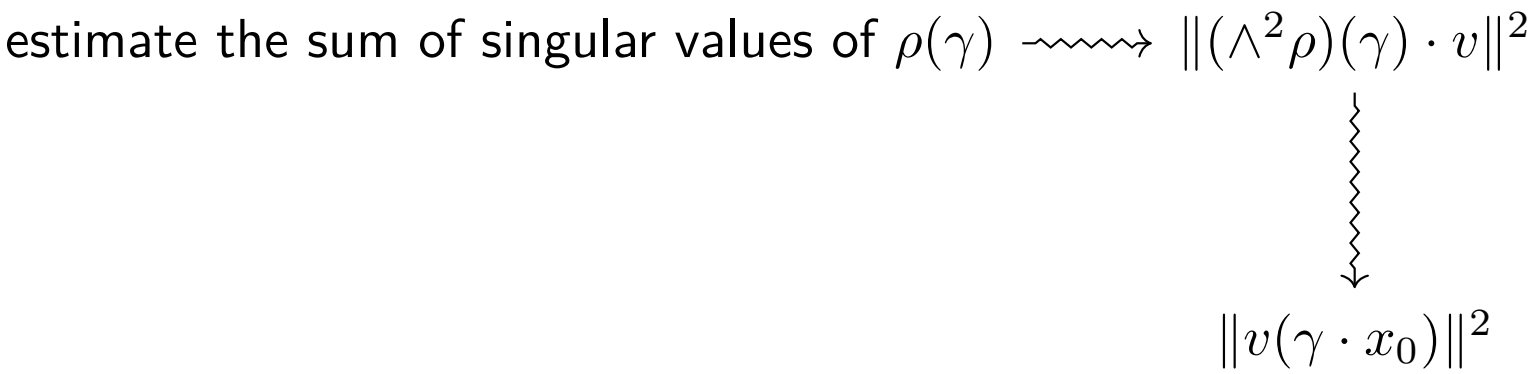

# Sketch of Proof

The core idea is to estimate the norm of the flat section with respect to the associated Higgs bundle $(\wedge^2\mathcal{E}, \wedge^2\Phi)$.

$$\text{estimate the sum of singular values of } \rho(\gamma) \rightsquigarrow \|(\wedge^2\rho)(\gamma)\cdot v\|^2 \downsquigarrow \|v(\gamma\cdot x_0)\|^2$$

It suffices to show that there exist constants $C_1, C_2, \varepsilon > 0$ such that

$$\|v(x)\|^2 \geqslant C_1 \cdot \exp(\varepsilon \cdot d_{\mathbb{H}^2}(x, x_0)) - C_2,$$

then the orbit map of $\rho$ is a quasi-isometric embedding.

# Filip's estimate (simplified)

**Lemma (Filip, 2021+)**

*Given a complete Riemannian manifold $(M, g)$ with the distance function $d\colon M \times M \to \mathbb{R}$ and a smooth function $f \in C^\infty(M;\mathbb{R})$. Suppose that $f$ satisfies*

(S1) *$f$ is a non-negative function with all zeroes isolated;*

(S2) $\|\mathrm{d}f\|_g \gtrsim f$;

(S3) $\|\mathrm{d}f\|_g \gtrsim f^{1/2}$.

*Then $f$ achieves its unique minimum at $x_{\min}$ and there exist constants $C_1, C_2, \varepsilon > 0$, such that*

$$f(x) \geqslant C_1 \cdot \exp(\varepsilon \cdot d(x_{\min}, x)) - C_2, \forall x \in M.$$

Recall that it suffices to deal with the polystable $4$-cyclic Higgs bundle

$$(\mathcal{E},\Phi)= \quad \mathcal{E}_0 \xrightarrow{\mathbf{1}} \mathcal{E}_1 \xrightarrow{\alpha} \mathcal{E}_2 \xrightarrow{\beta} \mathcal{E}_3 \xrightarrow{\gamma} \mathcal{E}_0$$

whose ranks $(\mathrm{rank}(\mathcal{E}_j))=(1,1,q,1)$ and signatures $(-,+,-,+)$.

Part of $(\wedge^2\mathcal{E}, \wedge^2\Phi)$ is

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# Thank you!