4-cyclic Higgs bundles, ∂-alternating surfaces and QIE representations

(in progress, joint with 李琼玲 Qiongling Li)

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南開大學
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harmonic map $\Longleftrightarrow$ Higgs bundle

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Goal: Study the monodromy via harmonic map.

Contents

1. Maximal spacelike immersion in $\mathbb{H}^{2,q}$
2. ∂-alternating surfaces in $\mathbb{CH}^{m,n}$
3. Bi-Lipschitz embedding & domain of discontinuity

Maximal spacelike immersion in $\mathbb{H}^{2,q}$

Pseudo-hyperbolic space

Let

$$\mathbb{R}^{2,q+1} := (\mathbb{R}^{2+(q+1)}, \mathbf{b}),$$

where

$$\mathbf{b}(v,w) = v_1 w_1 + v_2 w_2 - \sum_{j=3}^{q+3} v_j w_j$$

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The bilinear form $\mathbf{b}$ induces the pseudo-Riemannian metric $g_{\mathbb{H}^{2,q}}$ on $\mathbb{H}^{2,q}$ with signature $(2,q)$.

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A spacelike immersion f defines the following decomposition

$$\begin{array}{ccccccc} \mathbb{R}^{2,q+1} & = & U_p & \oplus & \ell_p & \oplus & N_p \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \text{tangent 2-plane} & & \mathbb{R}\cdot f(p) & & \text{normal space} \\ \text{sign} & & + & & - & & - \end{array}$$

at every $p \in X$.

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- $\mathbf{b}$ extends holomorphically on $\mathcal{U}$, $\mathcal{L}$, $\mathcal{N}$. In particular, $\mathcal{L} \cong \mathcal{L}^{-1}$, $\mathcal{U} \cong \mathcal{U}^\vee$, $\mathcal{N} \cong \mathcal{N}^\vee$.

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$$\mathrm{d}f = \partial f + \bar{\partial} f,$$

where

$$\partial f\colon T^{1,0}X \to \mathrm{Hom}(\ell \otimes \mathbb{C}, \mathcal{U}^{1,0}) \text{ and } \bar{\partial} f\colon T^{0,1}X \to \mathrm{Hom}(\ell \otimes \mathbb{C}, \mathcal{U}^{0,1})$$

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Therefore, $\mathcal{U}^{1,0} \cong \mathcal{LK}_X^{-1}$ and $\mathcal{U}^{0,1} \cong \mathcal{LK}_X$.

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The mean curvature of f vanishes implies that

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does not has $(1,1)$-part, where

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Then, $\mathbb{H}^{2,q}$ has constant scalar curvature implies that, by Codazzi equation,

$\mathrm{II}^{2,0}$ is a holomorphic section of $\mathrm{Hom}(\mathcal{K}_X^{-1}, \mathrm{Hom}(\mathcal{U}^{1,0}, \mathcal{N})) \cong \mathcal{L}\mathcal{K}_X^2\mathcal{N}$.

Also, $\mathrm{II}^{0,2}$ is holomorphic.

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$$\mathrm{D} = \nabla^H + \underbrace{\begin{pmatrix} & \partial f & \\ & & \mathrm{II}^{0,2} \\ \bar{\partial} f & & \\ \mathrm{II}^{2,0} & & \end{pmatrix}}_{\Phi} - \Phi^{*_H}$$

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Changing the sign of H on $\mathcal{L} \oplus \mathcal{N}$, we obtain an Hermitian metric h on $\underline{\mathbb{R}^{2,q+1}} \otimes \mathbb{C}$. Then

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$$\begin{cases} (\nabla^h)^{0,1}\Phi = 0, \\ F(\mathrm{D}) = 0 \implies F(\nabla^h) + [\Phi \wedge \Phi^{*_h}] = 0. \end{cases}$$

In other words, h is the harmonic metric of the following 4-cyclic Higgs bundle.

$$(\mathcal{E}_f, \Phi_f) := \begin{array}{ccccc} \mathcal{L}\mathcal{K}_X & \xrightarrow{\mathbf{1}} & (\mathcal{L}, \mathbf{b}|_{\mathcal{L}}) & \xrightarrow{\mathbf{1}} & \mathcal{L}\mathcal{K}_X^{-1} \\ & \nwarrow^{\beta} & & \swarrow_{\beta} & \\ & & (\mathcal{N}, \mathbf{b}|_{\mathcal{N}}) & & \end{array}$$

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where f is a ρ-equivariant maximal spacelike immersion, then $(\mathcal{E}_f, \Phi_f)$ descends to X and

$$\rho = \mathrm{NAH}(\mathcal{E}_f, \Phi_f)$$

through the non-Abelian Hodge correspondence.

Anosov property and domain of discontinuity

Theorem (Collier–Tholozan–Toulisse, 2019)

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Remark

Anosov property comes from the maximality of the Toledo invariant, not from the harmonic map.

∂-alternating surfaces in $\mathbb{CH}^{m,n}$

Complex pseudo-hyperbolic space

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The Hermitian form H induces the pseudo-Kähler metric H_0 on $\mathbb{CH}^{m,n}$ with signature (m,n).

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∂-alternating $\Longrightarrow$ spacelike in $(1,0)$-direction.

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- f is conformal $\implies$ $q_2 := \delta_f(\partial^{1,0}f)$ vanishes;

Set

$$L_f := f^*\mathcal{O}(-1), \quad V_f = f^*T^{1,0}\mathbb{CH}^{m,n} \cong \mathrm{Hom}(L_f, L_f^{\perp_H}).$$

We consider the $(1,0)$-part $\mathrm{d}^{1,0}f$ of $\mathrm{d}f$, which sends TX to V_f. It further decomposes as

$$\partial^{1,0}f \in \Omega^{1,0}(V_f) \quad \text{and} \quad \bar{\partial}^{1,0}f \in \Omega^{0,1}(V_f).$$

Let T^+ be the subbundle generated by $\partial^{1,0}f$.

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- f is conformal $\implies$ $q_2 := \delta_f(\partial^{1,0}f)$ vanishes;
- f is harmonic and conformal $\implies$ $\sigma^{2,0}$ is the usual $(2,0)$-part of the second fundamental form.

∂-alternating surfaces

A ∂-alternating surface $f\colon X \to \mathbb{CH}^{m,n}$ is a specific conformal and harmonic immersion with positive $T^+ = \langle \partial^{1,0} f \rangle$.

Proposition (Li–Z., 2026+)

Let $f\colon X \to \mathbb{CH}^{m,n}$ be a ∂-alternating surface. Then the trivial bundle $\underline{\mathbb{C}^{m,n+1}}$ over X admits a smooth H-orthorgonal decomposition

$$\underline{\mathbb{C}^{m,n+1}} = L_f \oplus T_+ \oplus N_- \oplus N_+,$$

where N_- is negative and N_+ is positive, such that the trivial connection

$$\mathrm{D} = \nabla^H + \underbrace{\begin{pmatrix} & & \gamma \\ \mathbf{1} & & \\ & \alpha & \\ & & \beta \end{pmatrix}}_{\Phi} - \Phi^{*_H}$$

and $(\nabla^H)^{0,1}\Phi = 0$. Here

$$\mathbf{1} = \partial^{1,0} f, \quad \alpha = \sigma^{2,0}, \quad \beta = ((\nabla^{V_f})^{1,0}|_{N_-})^{N^+} \text{ and } \gamma = \delta_f.$$

Again, by changing the signs of H on L_f and N_-, we obtain a harmonic metric h for the 4-cyclic Higgs bundle of the following form over X.

$$(\mathcal{E}_f, \Phi_f) = \mathcal{L} \xrightarrow[\mathbf{1}]{} \mathcal{L}\mathcal{K}_X^{-1} \xrightarrow[\alpha]{} \mathcal{V} \xrightarrow[\beta]{} \mathcal{W} \xrightarrow{\gamma} \mathcal{L}$$

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Proposition (Li–Z., 2026+)

Every harmonic 4-cyclic Higgs bundle of the form above defines a ∂-alternating surface on X.

Special cases

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- $\alpha, \beta \equiv 0 \implies f$ is a minimal immersion in $\mathbb{CH}^{m,0}$;
- $f\colon X \to \mathbb{H}^{2,q}$ spacelike conformal immersion, then

$$f \text{ is } \partial\text{-alternating in } \mathbb{CH}^{2,q} \iff f \text{ is maximal.}$$

Equivariant case

Let X be a closed Riemann surface whose genus $\geqslant 2$.

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$$\left(\rho\colon \pi_1(X) \to \mathrm{PU}_{m,n+1}, \quad f\colon \widetilde{X} \to \mathbb{CH}^{m,n}\right)$$

where f is a ρ-equivariant ∂-alternating surface, then $(\mathcal{E}_f, \Phi_f)$ descends to a polystable 4-cyclic $\mathrm{U}_{m,n+1}$-Higgs bundle over X and

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Theorem (Li–Z., 2026+)

The generalized Gauss map

$$\begin{aligned} \mathcal{G}_f\colon \widetilde{X} &\longrightarrow \{\textit{positive } m\textit{-planes in } \mathbb{C}^{m,n+1}\} = \mathrm{Sym}(\mathrm{PU}_{m,n+1}) \\ x &\longmapsto (T_{+,x} \oplus N_{+,x}) \end{aligned}$$

is the associated harmonic map of $(\mathcal{E}_f, \Phi_f)$.

Bi-Lipschitz embedding & domain of discontinuity

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- *The orbit map*

$$\tau_\rho \colon \pi_1(X) \longrightarrow \mathrm{Sym}(\mathrm{PU}_{2,q+1})$$
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Domain of discontinuity

Let $\mathcal{S}_i(\mathbb{C}^{m,n+1}) \subset \mathrm{Gr}_2(\mathbb{C}^{m,n+1})$ denote the space consisting of positive semi-defnite 2-planes with signature $(i, 0|2-i)$.

Theorem (Li–Z., 2026+)

Let (ρ, f) be an equivariant ∂-alternating surface to $\mathbb{CH}^{2,q}$ on a closed Riemann surface X whose genus $\geqslant 2$ and the corresponding β has a zero.

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Let (ρ, f) be an equivariant ∂-alternating surface to $\mathbb{CH}^{2,q}$ on a closed Riemann surface X whose genus $\geqslant 2$ and the corresponding β has a zero. Then for $i = 0, 1, 2$, the fibre bundle $\mathcal{F}_i$ over X whose fibre at $x \in X$ is

$$\{P_x \in \mathcal{S}_i(\mathbb{C}^{2,q+1}) \mid P_x \cap (T_{+,x} \oplus N_{-,x}) \neq \{0\}\}$$

is diffeomorphic to

$$\rho(\pi_1(X))\backslash\Omega_i$$

for open domains $\Omega_i \subset \mathcal{S}_i(\mathbb{C}^{2,q+1})$, and moreover $\Omega_2 = \mathcal{S}_2(\mathbb{C}^{2,q+1})$.

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for open domains $\Omega_i \subset \mathcal{S}_i(\mathbb{C}^{2,q+1})$, and moreover $\Omega_2 = \mathcal{S}_2(\mathbb{C}^{2,q+1})$. Furthermore, the diffeomorphic types of the fibre F_i of $\mathcal{F}_i$ are

$$F_0 \cong S^{2q-1} \times S^{2q-1}, \quad F_1 \cong S^{4q+1} \setminus (S^{2q-1} \times S^{2q-1}), \quad F_2 \cong \mathbb{R}^{4q+2}.$$

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Sketch of Proof

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$$\text{estimate the sum of singular values of } \rho(\gamma) \rightsquigarrow \|(\wedge^2\rho)(\gamma)\cdot v\|^2 \downsquigarrow \|v(\gamma\cdot x_0)\|^2$$

It suffices to show that there exist constants $C_1, C_2, \varepsilon > 0$ such that

$$\|v(x)\|^2 \geqslant C_1 \cdot \exp(\varepsilon \cdot d_{\mathbb{H}^2}(x, x_0)) - C_2,$$

then the orbit map of ρ is a quasi-isometric embedding.

Filip's estimate (simplified)

Lemma (Filip, 2021+)

Given a complete Riemannian manifold (M, g) with the distance function $d\colon M \times M \to \mathbb{R}$ and a smooth function $f \in C^\infty(M;\mathbb{R})$. Suppose that f satisfies

(S1) *f is a non-negative function with all zeroes isolated;*

(S2) $\|\mathrm{d}f\|_g \gtrsim f$;

(S3) $\|\mathrm{d}f\|_g \gtrsim f^{1/2}$.

Then f achieves its unique minimum at $x_{\min}$ and there exist constants $C_1, C_2, \varepsilon > 0$, such that

$$f(x) \geqslant C_1 \cdot \exp(\varepsilon \cdot d(x_{\min}, x)) - C_2, \forall x \in M.$$

Recall that it suffices to deal with the polystable 4-cyclic Higgs bundle

$$(\mathcal{E},\Phi)= \quad \mathcal{E}_0 \xrightarrow{\mathbf{1}} \mathcal{E}_1 \xrightarrow{\alpha} \mathcal{E}_2 \xrightarrow{\beta} \mathcal{E}_3 \xrightarrow{\gamma} \mathcal{E}_0$$

whose ranks $(\mathrm{rank}(\mathcal{E}_j))=(1,1,q,1)$ and signatures $(-,+,-,+)$.

Part of $(\wedge^2\mathcal{E}, \wedge^2\Phi)$ is

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Thank you!