Infinitesimal Rigidity of Cyclic Surfaces

(in progress, joint with 李琼玲 Qiongling Li)

张峻铭 Junming Zhang

Chern Institute of Mathematics, Nankai University

2026.01.14 at Nanjing University, Nanjing



Contents

1. Motivation
2. Cyclic Higgs Bundles and Cyclic Surfaces
3. Infinitesimal Rigidity

Motivation

Classical Teichmüller Theory

Let S be a closed connected oriented surface whose genus is larger than 2. Its **Teichmüller space** $\mathcal{T}(S)$ is the moduli space of marked hyperbolic structure over S.

Classical Teichmüller Theory

Let S be a closed connected oriented surface whose genus is larger than 2. Its **Teichmüller space** $\mathcal{T}(S)$ is the moduli space of marked hyperbolic structure over S.

The **mapping class group**

$$\mathrm{Mod}(S) := \pi_0(\mathrm{Diff}_+(S))$$

acts on $\mathcal{T}(S)$ naturally.

Classical Teichmüller Theory

Let S be a closed connected oriented surface whose genus is larger than 2. Its **Teichmüller space** $\mathcal{T}(S)$ is the moduli space of marked hyperbolic structure over S.

The **mapping class group**

$$\mathrm{Mod}(S) := \pi_0(\mathrm{Diff}_+(S))$$

acts on $\mathcal{T}(S)$ naturally.

The holonomy representations corresponding to the point in $\mathcal{T}(S)$ are called **Fuchsian** representations. Furthermore, we can view $\mathcal{T}(S)$ as a connected component of the character variety

$$\mathrm{Hom}(\pi_1(S), \mathrm{PSL}_2\mathbb{R})/\!/\mathrm{PSL}_2\mathbb{R}.$$

Move to Higher Rank

Instead of focusing on representations from $\pi_1(S)$ into $\mathrm{PSL}_2\mathbb{R}$, we replace the target by a semi-simple Lie group $G_{\mathbb{R}}$ of higher rank, such as

Move to Higher Rank

Instead of focusing on representations from $\pi_1(S)$ into $\mathrm{PSL}_2\mathbb{R}$, we replace the target by a semi-simple Lie group $G_{\mathbb{R}}$ of higher rank, such as

$$\mathrm{SL}_n\mathbb{R} \quad (n \geqslant 3),$$

Move to Higher Rank

Instead of focusing on representations from $\pi_1(S)$ into $\mathrm{PSL}_2\mathbb{R}$, we replace the target by a semi-simple Lie group $G_{\mathbb{R}}$ of higher rank, such as

$$\mathrm{SL}_n\mathbb{R} \quad (n \geqslant 3),$$

$$\mathrm{Sp}_{2n}\mathbb{R} \quad (n \geqslant 2),$$

Move to Higher Rank

Instead of focusing on representations from $\pi_1(S)$ into $\mathrm{PSL}_2\mathbb{R}$, we replace the target by a semi-simple Lie group $G_{\mathbb{R}}$ of higher rank, such as

$$\mathrm{SL}_n\mathbb{R} \quad (n \geqslant 3),$$

$$\mathrm{Sp}_{2n}\mathbb{R} \quad (n \geqslant 2),$$

$$\mathrm{SO}^0_{p,q} \quad (\min\{p,q\} \geqslant 2)$$

and so on.

Move to Higher Rank

Instead of focusing on representations from $\pi_1(S)$ into $\mathrm{PSL}_2\mathbb{R}$, we replace the target by a semi-simple Lie group $G_{\mathbb{R}}$ of higher rank, such as

$$\mathrm{SL}_n\mathbb{R} \quad (n \geqslant 3),$$

$$\mathrm{Sp}_{2n}\mathbb{R} \quad (n \geqslant 2),$$

$$\mathrm{SO}^0_{p,q} \quad (\min\{p,q\} \geqslant 2)$$

and so on.

In general, the set of discrete and faithful representations is only closed and NOT open.

Higher Teichmüller Spaces

Definition

A **higher Teichmüller space** is a subset of $\mathrm{Hom}(\pi_1(S), G_{\mathbb{R}})/\!/G_{\mathbb{R}}$, which is a union of connected components that consist entirely of discrete and faithful representations.

Higher Teichmüller Spaces

Definition

A **higher Teichmüller space** is a subset of $\mathrm{Hom}(\pi_1(S), G_{\mathbb{R}})/\!/G_{\mathbb{R}}$, which is a union of connected components that consist entirely of discrete and faithful representations.

We give a famous example of higher Teichmüller space here.

Example: Hitchin Component, in particular $G_{\mathbb{R}} = \mathrm{SL}_3\mathbb{R}$

Example (Hitchin, 1992; Labourie, 2006; Fock–Goncharov,2006)

Example: Hitchin Component, in particular $G_{\mathbb{R}} = \mathrm{SL}_3\mathbb{R}$

Example (Hitchin, 1992; Labourie, 2006; Fock–Goncharov,2006)

The component containing representations of the following form

$$\pi_1(S) \xrightarrow{\text{Fuchs}} \mathrm{PSL}_2\mathbb{R} \xrightarrow{\text{irr}} \mathrm{PSL}_3\mathbb{R} = \mathrm{SL}_3\mathbb{R}$$

is a higher Teichmüller space called the Hitchin component $\mathcal{T}_{Hit}(S, \mathrm{SL}_3\mathbb{R})$.

Example: Hitchin Component, in particular $G_{\mathbb{R}} = \mathrm{SL}_3\mathbb{R}$

Example (Hitchin, 1992; Labourie, 2006; Fock–Goncharov,2006)

The component containing representations of the following form

$$\pi_1(S) \xrightarrow{\text{Fuchs}} \mathrm{PSL}_2\mathbb{R} \xrightarrow{\text{irr}} \mathrm{PSL}_3\mathbb{R} = \mathrm{SL}_3\mathbb{R}$$

is a higher Teichmüller space called the Hitchin component $\mathcal{T}_{Hit}(S, \mathrm{SL}_3\mathbb{R})$. When fixing an arbitrary complex structure j on S, we have the parametrization

$$\mathcal{T}_{Hit}(S, \mathrm{SL}_3\mathbb{R}) \cong \mathrm{H}^0(X, \mathcal{K}_X^2) \oplus \mathrm{H}^0(X, \mathcal{K}_X^3),$$

where $\mathcal{K}_X$ denotes the holomorphic cotangent bundle of the Riemann surface $X = (S, \mathrm{j})$.

Example: Hitchin Component, in particular $G_{\mathbb{R}} = \mathrm{SL}_3\mathbb{R}$

Example (Hitchin, 1992; Labourie, 2006; Fock–Goncharov,2006)

The component containing representations of the following form

$$\pi_1(S) \xrightarrow{\text{Fuchs}} \mathrm{PSL}_2\mathbb{R} \xrightarrow{\text{irr}} \mathrm{PSL}_3\mathbb{R} = \mathrm{SL}_3\mathbb{R}$$

is a higher Teichmüller space called the Hitchin component $\mathcal{T}_{Hit}(S, \mathrm{SL}_3\mathbb{R})$. When fixing an arbitrary complex structure j on S, we have the parametrization

$$\mathcal{T}_{Hit}(S, \mathrm{SL}_3\mathbb{R}) \cong \mathrm{H}^0(X, \mathcal{K}_X^2) \oplus \mathrm{H}^0(X, \mathcal{K}_X^3),$$

where $\mathcal{K}_X$ denotes the holomorphic cotangent bundle of the Riemann surface $X = (S, \mathrm{j})$.

Remark

This parametrization does not behave well under the mapping class group action!

Mapping Class Group Equivariant Parametrization

Theorem (Loftin, 2001; Labourie, 2007; Labourie, 2017)

When $G_{\mathbb{R}}$ is a simple real split Lie group of rank 2, i.e. $\mathrm{SL}_3\mathbb{R}$, $\mathrm{PSp}_4\mathbb{R}$ or $G_2^{(2)}$, there exists an analytic diffeomorphism between the Hitchin component $\mathcal{T}_{Hit}(S, G_{\mathbb{R}})$ and

$$\left\{ (\mathrm{j}, Q) \;\middle|\; \mathrm{j} \in \mathcal{T}(S), Q \in \mathrm{H}^0\left(X, \mathcal{K}_X^{\dim(G_{\mathbb{R}})/2-1}\right), \text{ where } X = (S, \mathrm{j}) \right\}$$

equivariant under the mapping class group $\mathrm{Mod}(S)$ action.

Labourie's Conjecture

The key point of Labourie's proof is the following theorem:

Theorem (Labourie, 2007; Labourie, 2017)

When $G_{\mathbb{R}}$ is a simple real split Lie group of rank 2, for any $\rho \in \mathcal{T}_{Hit}(S, G_{\mathbb{R}})$, there exists a unique minimal map $f : \widetilde{S} \to G_{\mathbb{R}}/K_{\mathbb{R}}$, where $K_{\mathbb{R}}$ denotes the maximal compact subgroup of $G_{\mathbb{R}}$.

Labourie's Conjecture

The key point of Labourie's proof is the following theorem:

Theorem (Labourie, 2007; Labourie, 2017)

When $G_{\mathbb{R}}$ is a simple real split Lie group of rank 2, for any $\rho \in \mathcal{T}_{Hit}(S, G_{\mathbb{R}})$, there exists a unique minimal map $f : \widetilde{S} \to G_{\mathbb{R}}/K_{\mathbb{R}}$, where $K_{\mathbb{R}}$ denotes the maximal compact subgroup of $G_{\mathbb{R}}$.

Remark

When $G_{\mathbb{R}}$ is an arbitrary semisimple Lie group,

- $\exists$ *was proven by* [Labourie, 2008] *for general Anosov representations.*

Labourie's Conjecture

The key point of Labourie's proof is the following theorem:

Theorem (Labourie, 2007; Labourie, 2017)

When $G_{\mathbb{R}}$ is a simple real split Lie group of rank 2, for any $\rho \in \mathcal{T}_{Hit}(S, G_{\mathbb{R}})$, there exists a unique minimal map $f : \widetilde{S} \to G_{\mathbb{R}}/K_{\mathbb{R}}$, where $K_{\mathbb{R}}$ denotes the maximal compact subgroup of $G_{\mathbb{R}}$.

Remark

When $G_{\mathbb{R}}$ is an arbitrary semisimple Lie group,

- $\exists$ *was proven by* [Labourie, 2008] *for general Anosov representations.*
- ! *was conjectured to be true for any Hitchin representations (resp. higher Teichmüller spaces). (**Labourie's conjecture**)*

Labourie's Conjecture

When $\mathrm{rank}(G_{\mathbb{R}}) \geqslant 3$, ! fails for

Labourie's Conjecture

When $\mathrm{rank}(G_{\mathbb{R}}) \geqslant 3$, ! fails for

- some multi-Fuchsian representations $\pi_1(S) \to (\mathrm{PSL}_2\mathbb{R})^n$, where $n \geqslant 3$; [Markovic, 2022]

Labourie's Conjecture

When $\mathrm{rank}(G_{\mathbb{R}}) \geqslant 3$, ! fails for

- some multi-Fuchsian representations $\pi_1(S) \to (\mathrm{PSL}_2\mathbb{R})^n$, where $n \geqslant 3$; [Markovic, 2022]
- some Hitchin representations $\pi_1(S) \to G_{\mathbb{R}}$, where $G_{\mathbb{R}}$ is real split; [Sagman–Smillie, 2022+]

Labourie's Conjecture

When $\mathrm{rank}(G_{\mathbb{R}}) \geqslant 3$, ! fails for

- some multi-Fuchsian representations $\pi_1(S) \to (\mathrm{PSL}_2\mathbb{R})^n$, where $n \geqslant 3$; [Markovic, 2022]
- some Hitchin representations $\pi_1(S) \to G_{\mathbb{R}}$, where $G_{\mathbb{R}}$ is real split; [Sagman–Smillie, 2022+]

When $\mathrm{rank}_{\mathbb{R}}(G_{\mathbb{R}}) = 2$, Labourie's conjecture was proven for

Labourie's Conjecture

When $\mathrm{rank}(G_{\mathbb{R}}) \geqslant 3$, ! fails for

- some multi-Fuchsian representations $\pi_1(S) \to (\mathrm{PSL}_2\mathbb{R})^n$, where $n \geqslant 3$; [Markovic, 2022]
- some Hitchin representations $\pi_1(S) \to G_{\mathbb{R}}$, where $G_{\mathbb{R}}$ is real split; [Sagman–Smillie, 2022+]

When $\mathrm{rank}_{\mathbb{R}}(G_{\mathbb{R}}) = 2$, Labourie's conjecture was proven for

(1) any multi-Fuchsian representation $\pi_1(S) \to (\mathrm{PSL}_2\mathbb{R})^2$; [Schoen, 1993; Bosante–Schlenker, 2010]

Labourie's Conjecture

When $\mathrm{rank}(G_{\mathbb{R}}) \geqslant 3$, ! fails for

- some multi-Fuchsian representations $\pi_1(S) \to (\mathrm{PSL}_2\mathbb{R})^n$, where $n \geqslant 3$; [Markovic, 2022]
- some Hitchin representations $\pi_1(S) \to G_{\mathbb{R}}$, where $G_{\mathbb{R}}$ is real split; [Sagman–Smillie, 2022+]

When $\mathrm{rank}_{\mathbb{R}}(G_{\mathbb{R}}) = 2$, Labourie's conjecture was proven for

(1) any multi-Fuchsian representation $\pi_1(S) \to (\mathrm{PSL}_2\mathbb{R})^2$; [Schoen, 1993; Bosante–Schlenker, 2010]

(2) any maximal representation $\pi_1(S) \to \mathrm{PSp}_4\mathbb{R}$; [Collier, 2016; Alessandrini–Collier, 2018]

Labourie's Conjecture

When $\mathrm{rank}(G_{\mathbb{R}}) \geqslant 3$, ! fails for

- some multi-Fuchsian representations $\pi_1(S) \to (\mathrm{PSL}_2\mathbb{R})^n$, where $n \geqslant 3$; [Markovic, 2022]
- some Hitchin representations $\pi_1(S) \to G_{\mathbb{R}}$, where $G_{\mathbb{R}}$ is real split; [Sagman–Smillie, 2022+]

When $\mathrm{rank}_{\mathbb{R}}(G_{\mathbb{R}}) = 2$, Labourie's conjecture was proven for

(1) any multi-Fuchsian representation $\pi_1(S) \to (\mathrm{PSL}_2\mathbb{R})^2$; [Schoen, 1993; Bosante–Schlenker, 2010]

(2) any maximal representation $\pi_1(S) \to \mathrm{PSp}_4\mathbb{R}$; [Collier, 2016; Alessandrini–Collier, 2018]

(3) any maximal representation $\pi_1(S) \to \mathrm{SO}^0_{2,n+1}$; [Collier–Tholozan–Toulisse, 2019]

Labourie's Conjecture

When $\mathrm{rank}(G_{\mathbb{R}}) \geqslant 3$, ! fails for

- some multi-Fuchsian representations $\pi_1(S) \to (\mathrm{PSL}_2\mathbb{R})^n$, where $n \geqslant 3$; [Markovic, 2022]
- some Hitchin representations $\pi_1(S) \to G_{\mathbb{R}}$, where $G_{\mathbb{R}}$ is real split; [Sagman–Smillie, 2022+]

When $\mathrm{rank}_{\mathbb{R}}(G_{\mathbb{R}}) = 2$, Labourie's conjecture was proven for

(1) any multi-Fuchsian representation $\pi_1(S) \to (\mathrm{PSL}_2\mathbb{R})^2$; [Schoen, 1993; Bosante–Schlenker, 2010]

(2) any maximal representation $\pi_1(S) \to \mathrm{PSp}_4\mathbb{R}$; [Collier, 2016; Alessandrini–Collier, 2018]

(3) any maximal representation $\pi_1(S) \to \mathrm{SO}^0_{2,n+1}$; [Collier–Tholozan–Toulisse, 2019]

Remark

(2) uses Labourie's method; (1)(3) use more geometric method involving the geometry of $\mathbb{H}^{2,n}$.

Infinitesimal Rigidity

Key of Labourie's method: No "non-tangential" first order deformations (Jacobi fields) of some holomorphic curves tangent to a distribution in a homogeneous space of $G_{\mathbb{C}}$ (cyclic surfaces), which are **lifting of minimal surfaces in the symmetric space of** $G_{\mathbb{R}}$.

Infinitesimal Rigidity

Key of Labourie's method: No "non-tangential" first order deformations (Jacobi fields) of some holomorphic curves tangent to a distribution in a homogeneous space of $G_{\mathbb{C}}$ (cyclic surfaces), which are **lifting of minimal surfaces in the symmetric space of** $G_{\mathbb{R}}$.

Such infinitesimal rigidity was also proven for **alternating curves**.

Alternating Curves

Definition (Nie, 2024; Collier–Toulisse, 2024 for $\mathbb{H}^{4,2}$)

Let $M = \mathbb{H}^{n,n}$ or $\mathbb{H}^{n+1,n-1}$. A smooth embbeded spacelike surface $\Sigma \hookrightarrow M$ equipped with the induced metric $\langle \bullet, \bullet \rangle$ and the Levi-Civita connection ∇ on $\mathrm{T}M|_\Sigma$ is called an **alternating curve** or an **A-surface** if $\mathrm{T}M|_\Sigma$ has an orthogonal splitting of rank 2 real vector bundles:

$$\mathrm{T}M|_\Sigma = L_1 \oplus L_2 \oplus \cdots \oplus L_n$$

such that

Alternating Curves

Definition (Nie, 2024; Collier–Toulisse, 2024 for $\mathbb{H}^{4,2}$)

Let $M = \mathbb{H}^{n,n}$ or $\mathbb{H}^{n+1,n-1}$. A smooth embbeded spacelike surface $\Sigma \hookrightarrow M$ equipped with the induced metric $\langle \bullet, \bullet \rangle$ and the Levi-Civita connection ∇ on $\mathrm{T}M|_\Sigma$ is called an **alternating curve** or an **A-surface** if $\mathrm{T}M|_\Sigma$ has an orthogonal splitting of rank 2 real vector bundles:

$$\mathrm{T}M|_\Sigma = L_1 \oplus L_2 \oplus \cdots \oplus L_n$$

such that

(1) $L_1 \cong \mathrm{T}\Sigma$;

(2) $\langle \bullet, \bullet \rangle|_{L_i}$ are alternatively positive definite or negative definite;

(3) $(\nabla_X \xi_j)^{L_i} = 0$, $\forall X \in C^\infty(\Sigma, \mathrm{T}\Sigma), \xi_j \in C^\infty(\Sigma, L_j)$ where $|i-j| \geqslant 2$;

(4) $\alpha_i(X) := (\nabla_X \bullet)^{L_i} : L_{i-1} \to L_i$ is conformal for any $2 \leqslant i \leqslant n-1$;

(5) Σ is minimal.

Theorem (Nie, 2024)

Given a representation $\rho\colon \pi_1(\Sigma) \to \mathrm{SO}^0_{n,n+1}$ *and a* ρ*-equivariant alternating curve* $f : \widetilde{\Sigma} \to M$*. If* $Z(\alpha_2) \leqslant Z(\alpha_3) \leqslant \cdots \leqslant Z(\alpha_n)$*, then there is no nonzero Jacobi field of* f *in normal direction with compact support on* Σ*.*

Theorem (Nie, 2024)

Given a representation $\rho\colon \pi_1(\Sigma) \to \mathrm{SO}^0_{n,n+1}$ and a ρ-equivariant alternating curve $f : \widetilde{\Sigma} \to M$. If $Z(\alpha_2) \leqslant Z(\alpha_3) \leqslant \cdots \leqslant Z(\alpha_n)$, then there is no nonzero Jacobi field of f in normal direction with compact support on Σ.

Remark

Nie made use of the comparison among the norms of α_i's, which is equivalent to use the Hitchin equation of ***cyclic*** $\mathrm{SO}^0_{n,n+1}$*-**Higgs bundles**.*

Theorem (Nie, 2024)

Given a representation $\rho\colon \pi_1(\Sigma) \to \mathrm{SO}^0_{n,n+1}$ and a ρ-equivariant alternating curve $f : \widetilde{\Sigma} \to M$. If $Z(\alpha_2) \leqslant Z(\alpha_3) \leqslant \cdots \leqslant Z(\alpha_n)$, then there is no nonzero Jacobi field of f in normal direction with compact support on Σ.

Remark

Nie made use of the comparison among the norms of α_i's, which is equivalent to use the Hitchin equation of ***cyclic*** $\mathrm{SO}^0_{n,n+1}$*-**Higgs bundles**.*

Theorem (Collier–Toulisse, 2024)

Let Σ be a closed surface. Given a representation $\rho\colon \pi_1(\Sigma) \to G_2^{(2)}$ and a ρ-equivariant alternating curve $f : \widetilde{\Sigma} \to \mathbb{H}^{4,2}$. If $Z(\alpha_3) = 0$, then there is no nonzero Jacobi field of f in normal direction on Σ.

Theorem (Nie, 2024)

Given a representation $\rho\colon \pi_1(\Sigma) \to \mathrm{SO}^0_{n,n+1}$ and a ρ-equivariant alternating curve $f : \widetilde{\Sigma} \to M$. If $Z(\alpha_2) \leqslant Z(\alpha_3) \leqslant \cdots \leqslant Z(\alpha_n)$, then there is no nonzero Jacobi field of f in normal direction with compact support on Σ.

Remark

Nie made use of the comparison among the norms of α_i's, which is equivalent to use the Hitchin equation of ***cyclic*** $\mathrm{SO}^0_{n,n+1}$*-**Higgs bundles**.*

Theorem (Collier–Toulisse, 2024)

Let Σ be a closed surface. Given a representation $\rho\colon \pi_1(\Sigma) \to G_2^{(2)}$ and a ρ-equivariant alternating curve $f : \widetilde{\Sigma} \to \mathbb{H}^{4,2}$. If $Z(\alpha_3) = 0$, then there is no nonzero Jacobi field of f in normal direction on Σ.

Remark

- *Collier–Toulisse made use of Labourie's method;*
- *The intersection of two results above is the Hitchin representation to $G_2^{(2)}$.*

Remark

There is an analogue of Nie's result when replacing $\mathbb{H}^{n,n}$ *or* $\mathbb{H}^{n+1,n-1}$ *by para-complex hyperbolic space. Then the alternating curves are corresponding to the cyclic* $\mathrm{SL}_{2n+1}\mathbb{R}$*-Higgs bundles.* [Rungi–Tamburelli, 2025+]

Remark

There is an analogue of Nie's result when replacing $\mathbb{H}^{n,n}$ *or* $\mathbb{H}^{n+1,n-1}$ *by para-complex hyperbolic space. Then the alternating curves are corresponding to the cyclic* $\mathrm{SL}_{2n+1}\mathbb{R}$*-Higgs bundles.* [Rungi–Tamburelli, 2025+]

Goal: Give a unified approach of all these results about infinitesimal rigidity.

Cyclic Higgs Bundles and Cyclic Surfaces

Higgs bundles

The Higgs bundle is a useful tool to study the higher Teichmüller space. It is usually used to give a parametrization of the higher Teichmüller space. We fix a complex structure j on a smooth oriented surface Σ (not assumed to be closed) such that it becomes a Riemann surface X. Let $\mathcal{K}_X$ be its canonical line bundle.

Higgs bundles

The Higgs bundle is a useful tool to study the higher Teichmüller space. It is usually used to give a parametrization of the higher Teichmüller space. We fix a complex structure j on a smooth oriented surface Σ (not assumed to be closed) such that it becomes a Riemann surface X. Let $\mathcal{K}_X$ be its canonical line bundle.

Definition

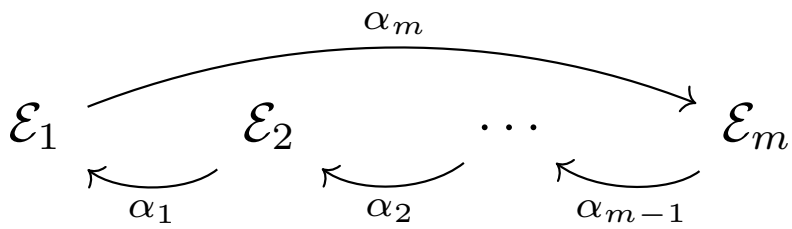
A **$(\mathrm{GL}_n\mathbb{C})$-Higgs bundle** over X is a pair $(\mathcal{E}, \varphi)$ consisting of the following data:

- a holomorphic vector bundle $\mathcal{E}$ over X with $\mathrm{rank}(\mathcal{E}) = n$;
- a holomorphic section $\varphi \in \mathrm{H}^0(X, \mathrm{End}(\mathcal{E}) \otimes \mathcal{K}_X)$ called **Higgs field**.

$(\mathcal{E}, \varphi)$ is furthermore an $\mathrm{SL}_n\mathbb{C}$-Higgs bundle if $\det(\mathcal{E}) = \mathcal{O}_X$ and $\mathrm{tr}(\varphi) = 0$.

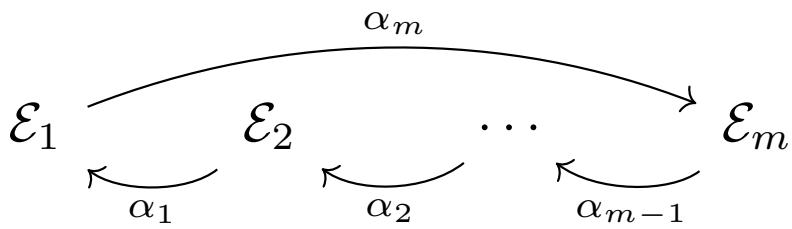
Definition

$(\mathcal{E}, \varphi)$ is called **cyclic** if it is of the form:



Definition

$(\mathcal{E}, \varphi)$ is called **cyclic** if it is of the form:



Definition

A **harmonic metric** of $(\mathcal{E}, \varphi)$ is an Hermitian metric h on $\mathcal{E}$ satisfying Hitchin equation

$$F(\nabla^h) + [\varphi, \varphi^{*_h}] = 0, \tag{1}$$

where ∇^h is the Chern connection of the metric h, $F(\nabla^h)$ denotes its curvature form and φ^{*_h} is the adjoint of φ with respect to h. Equivalently, h is harmonic if and only if $\nabla^h + \varphi + \varphi^{*_h}$ is flat.
We call $(\mathcal{E}, \varphi, h)$ a harmonic Higgs bundle if h is a harmonic metric of $(\mathcal{E}, \varphi)$.

A harmonic $\mathrm{SL}_n\mathbb{C}$-Higgs bundle $(\mathcal{E}, \varphi, h)$ over $X = (\Sigma, \mathrm{j})$ is equivalent to the pair

$$(\rho\colon \pi_1(\Sigma) \to \mathrm{SL}_n\mathbb{C}, h\colon \widetilde{\Sigma} \to \mathrm{SL}_n\mathbb{C}/\mathrm{SU}_n),$$

where h is a ρ-equivariant harmonic map.

A harmonic $\mathrm{SL}_n\mathbb{C}$-Higgs bundle $(\mathcal{E}, \varphi, h)$ over $X = (\Sigma, \mathrm{j})$ is equivalent to the pair

$$(\rho\colon \pi_1(\Sigma) \to \mathrm{SL}_n\mathbb{C}, h\colon \widetilde{\Sigma} \to \mathrm{SL}_n\mathbb{C}/\mathrm{SU}_n),$$

where h is a ρ-equivariant harmonic map.

When $(\mathcal{E}, \varphi)$ is cyclic, we say a harmonic metric is cyclic if $\mathcal{E}_j$'s are perpendicular to each other, then the harmonic map lifts to ρ-equivariant map

$$f\colon \widetilde{\Sigma} \to \mathrm{SL}_n\mathbb{C}/(\mathrm{SU}_{d_1} \times \cdots \times \mathrm{SU}_{d_m}),$$

where $d_j = \mathrm{rank}(\mathcal{E}_j)$. When $m \geqslant 3$, f is branched minimal.

When Σ is closed...

When Σ is closed, we have the following stability conditions.

Definition

We say a Higgs bundle $(\mathcal{E}, \varphi)$ where $\deg(\mathcal{E}) = 0$ is

- **stable** if $\deg(\mathcal{F}) < 0$ for any φ-invariant subbundle $\mathcal{F} \neq 0, \mathcal{E}$;

When Σ is closed...

When Σ is closed, we have the following stability conditions.

Definition

We say a Higgs bundle $(\mathcal{E}, \varphi)$ where $\deg(\mathcal{E}) = 0$ is

- **stable** if $\deg(\mathcal{F}) < 0$ for any φ-invariant subbundle $\mathcal{F} \neq 0, \mathcal{E}$;
- **polystable** if $(\mathcal{E}, \varphi)$ decomposes as $\bigoplus_i (\mathcal{F}_i, \varphi_i)$, where $(\mathcal{F}_i, \varphi_i)$ are stable Higgs bundles with $\deg(\mathcal{F}_i) = 0$.

When Σ is closed...

When Σ is closed, we have the following stability conditions.

Definition

We say a Higgs bundle $(\mathcal{E}, \varphi)$ where $\deg(\mathcal{E}) = 0$ is

- **stable** if $\deg(\mathcal{F}) < 0$ for any φ-invariant subbundle $\mathcal{F} \neq 0, \mathcal{E}$;
- **polystable** if $(\mathcal{E}, \varphi)$ decomposes as $\bigoplus_i (\mathcal{F}_i, \varphi_i)$, where $(\mathcal{F}_i, \varphi_i)$ are stable Higgs bundles with $\deg(\mathcal{F}_i) = 0$.

Theorem (Hitchin, 1987; Simpson, 1988)

A Higgs bundle $(\mathcal{E}, \Phi)$ with $\deg(\mathcal{E}) = 0$ is polystable if and only if there exists a harmonic metric h of $(\mathcal{E}, \Phi)$. Moreover, such metric is unique (up to scaling) if and only if $(\mathcal{E}, \Phi)$ is stable.

When Σ is closed...

When Σ is closed, we have the following stability conditions.

Definition

We say a Higgs bundle $(\mathcal{E}, \varphi)$ where $\deg(\mathcal{E}) = 0$ is

- **stable** if $\deg(\mathcal{F}) < 0$ for any φ-invariant subbundle $\mathcal{F} \neq 0, \mathcal{E}$;
- **polystable** if $(\mathcal{E}, \varphi)$ decomposes as $\bigoplus_i (\mathcal{F}_i, \varphi_i)$, where $(\mathcal{F}_i, \varphi_i)$ are stable Higgs bundles with $\deg(\mathcal{F}_i) = 0$.

Theorem (Hitchin, 1987; Simpson, 1988)

A Higgs bundle $(\mathcal{E}, \Phi)$ with $\deg(\mathcal{E}) = 0$ is polystable if and only if there exists a harmonic metric h of $(\mathcal{E}, \Phi)$. Moreover, such metric is unique (up to scaling) if and only if $(\mathcal{E}, \Phi)$ is stable.

Remark

When $(\mathcal{E}, \Phi)$ is stable and cyclic, the unique harmonic metric is also cyclic.

Cyclic Space

Let $G = \mathrm{SL}_n\mathbb{C}$, $\mathbb{C}^n = \bigoplus_{j\in\mathbb{Z}/m\mathbb{Z}} V_j$ orthogonal, where $m \geqslant 3$. Then

$$\mathfrak{g}_k := \left(\bigoplus_{j\in\mathbb{Z}/m\mathbb{Z}} \mathrm{Hom}(V_{j+k}, V_j) \right)_0$$

defines a $\mathbb{Z}/m\mathbb{Z}$-grading on $\mathfrak{g} = \mathfrak{sl}_n\mathbb{C}$.
Define

$$K_0 := S\left(\prod_{j\in\mathbb{Z}/m\mathbb{Z}} \mathrm{U}(V_j) \right).$$

Definition

The **cyclic space** is defined as the homogeneous space $\mathbb{X} := G/K_0$.

We fix the maximal torus

$$\mathfrak{t} = \left\{ \operatorname{diag}(a_i) \;\middle|\; \sum_i a_i = 0 \right\}$$

and this gives the lowest root $\alpha\colon\ \operatorname{diag}(a_i) \mapsto a_n - a_1$.
When $d_1 = d_m = 1$, the root space $\mathfrak{g}_\alpha$ is H_0-invariant. Below we always add this assumption.

We fix the maximal torus

$$\mathfrak{t} = \left\{ \operatorname{diag}(a_i) \;\middle|\; \sum_i a_i = 0 \right\}$$

and this gives the lowest root $\alpha\colon\ \operatorname{diag}(a_i) \mapsto a_n - a_1$.
When $d_1 = d_m = 1$, the root space $\mathfrak{g}_\alpha$ is H_0-invariant. Below we always add this assumption.

For any H_0-vector space V, let $[V] := G \times_{H_0} V$ denote the vector bundle over $\mathbb{X}$ associated to the H_0-bundle $G \to \mathbb{X}$.

We fix the maximal torus

$$\mathfrak{t} = \left\{ \operatorname{diag}(a_i) \;\middle|\; \sum_i a_i = 0 \right\}$$

and this gives the lowest root $\alpha\colon\ \operatorname{diag}(a_i) \mapsto a_n - a_1$.
When $d_1 = d_m = 1$, the root space $\mathfrak{g}_\alpha$ is H_0-invariant. Below we always add this assumption.

For any H_0-vector space V, let $[V] := G \times_{H_0} V$ denote the vector bundle over $\mathbb{X}$ associated to the H_0-bundle $G \to \mathbb{X}$.

Let $\omega_{MC} \in \mathcal{A}^1(G, \mathfrak{g})$ be the Maurer–Cartan form. It decomposes as $A + \omega$, where A gives a connection on the H_0-bundle $G \to \mathbb{X}$ and ω descends to a 1-form on $\mathbb{X}$ with value in $[\mathfrak{h}_0^\perp]$, which identifies $\mathrm{T}\mathbb{X}$ with $[\mathfrak{h}_0^\perp]$.

We fix the maximal torus

$$\mathfrak{t} = \left\{ \operatorname{diag}(a_i) \;\middle|\; \sum_i a_i = 0 \right\}$$

and this gives the lowest root $\alpha\colon\ \operatorname{diag}(a_i) \mapsto a_n - a_1$.
When $d_1 = d_m = 1$, the root space $\mathfrak{g}_\alpha$ is H_0-invariant. Below we always add this assumption.

For any H_0-vector space V, let $[V] := G \times_{H_0} V$ denote the vector bundle over $\mathbb{X}$ associated to the H_0-bundle $G \to \mathbb{X}$.

Let $\omega_{MC} \in \mathcal{A}^1(G, \mathfrak{g})$ be the Maurer–Cartan form. It decomposes as $A + \omega$, where A gives a connection on the H_0-bundle $G \to \mathbb{X}$ and ω descends to a 1-form on $\mathbb{X}$ with value in $[\mathfrak{h}_0^\perp]$, which identifies $\mathrm{T}\mathbb{X}$ with $[\mathfrak{h}_0^\perp]$.
Define

$$\omega_\bullet := \mathrm{pr}_{\mathfrak{g}_\bullet}(\omega) \in \mathcal{A}^1(\mathbb{X}, [\mathfrak{g}_\bullet]).$$

Cyclic Surfaces

Definition

The **cyclic distribution** $\mathcal{D} \subset \mathbb{TX}$ is the fixed point of the conjugate transpose in $[\mathfrak{g}_1 \oplus \mathfrak{g}_{-1}]$. It has the almost complex structure $\mathcal{J}$ induced by $(x, y) \mapsto (ix, -iy)$.

Cyclic Surfaces

Definition

The **cyclic distribution** $\mathcal{D} \subset \mathrm{T}\mathbb{X}$ is the fixed point of the conjugate transpose in $[\mathfrak{g}_1 \oplus \mathfrak{g}_{-1}]$. It has the almost complex structure $\mathcal{J}$ induced by $(\mathrm{x}, \mathrm{y}) \mapsto (\mathrm{ix}, -\mathrm{iy})$.

Definition

Let Σ be a smooth surface. $(\rho\colon \pi_1(\Sigma) \to \mathrm{SL}_n\mathbb{C}, f)$ is called an equivariant α-cyclic surface if $f : \widetilde{\Sigma} \to \mathbb{X}$ is a smooth ρ-equivariant map satisfying that

- f is tangent to $\mathcal{D}$;
- $\mathrm{d}f(\mathrm{T}\widetilde{\Sigma})$ is $\mathcal{J}$-invariant;
- $f^*(\omega_\alpha) \in \mathcal{A}^1(\widetilde{\Sigma}, f^*[\mathfrak{g}_\alpha])$ is nowhere vanishing.

Let $\pi : \widetilde{\Sigma} \to \Sigma$ be the universal cover.

Let $\pi : \widetilde{\Sigma} \to \Sigma$ be the universal cover.

Proposition

Given an equivariant α-cyclic surface (ρ, f) from Σ. Then f is an immersion and there exists a unique complex structure j *on Σ such that*

Let $\pi\colon \widetilde{\Sigma} \to \Sigma$ be the universal cover.

Proposition

Given an equivariant α-cyclic surface (ρ, f) from Σ. Then f is an immersion and there exists a unique complex structure j *on Σ such that*

- $$(\mathcal{E} := \pi_* f^* G[\mathbb{C}^n], \varphi := \pi_* f^*(\omega_1))$$

  defines a cyclic $\mathrm{SL}_n\mathbb{C}$-Higgs bundle over (Σ, j);

Let $\pi\colon \widetilde{\Sigma} \to \Sigma$ be the universal cover.

Proposition

Given an equivariant α-cyclic surface (ρ, f) from Σ. Then f is an immersion and there exists a unique complex structure j *on Σ such that*

- $$(\mathcal{E} := \pi_* f^* G[\mathbb{C}^n], \varphi := \pi_* f^*(\omega_1))$$

  defines a cyclic $\mathrm{SL}_n\mathbb{C}$-Higgs bundle over (Σ, j);
- $\mathrm{pr}_{\mathfrak{g}_\alpha}(\varphi)$ *is nowhere vanishing;*

Let $\pi\colon \widetilde{\Sigma} \to \Sigma$ be the universal cover.

Proposition

Given an equivariant α-cyclic surface (ρ, f) from Σ. Then f is an immersion and there exists a unique complex structure j *on Σ such that*

- $$(\mathcal{E} := \pi_* f^* G[\mathbb{C}^n], \varphi := \pi_* f^*(\omega_1))$$

  defines a cyclic $\mathrm{SL}_n\mathbb{C}$-Higgs bundle over (Σ, j);
- $\mathrm{pr}_{\mathfrak{g}_\alpha}(\varphi)$ *is nowhere vanishing;*
- $h\colon \widetilde{\Sigma} \xrightarrow{f} \mathbb{X} \to \mathrm{SL}_n\mathbb{C}/\mathrm{SU}_n$ *is a ρ-equivariant minimal surface and induces a cyclic harmonic metric of $(\mathcal{E}, \varphi)$.*

Let $\pi\colon \widetilde{\Sigma} \to \Sigma$ be the universal cover.

Proposition

Given an equivariant α-cyclic surface (ρ, f) from Σ. Then f is an immersion and there exists a unique complex structure j on Σ such that

- $$(\mathcal{E} := \pi_* f^* G[\mathbb{C}^n], \varphi := \pi_* f^*(\omega_1))$$

  defines a cyclic $\mathrm{SL}_n\mathbb{C}$-Higgs bundle over (Σ, j);
- $\mathrm{pr}_{\mathfrak{g}_\alpha}(\varphi)$ *is nowhere vanishing;*
- $h\colon \widetilde{\Sigma} \xrightarrow{f} \mathbb{X} \to \mathrm{SL}_n\mathbb{C}/\mathrm{SU}_n$ *is a ρ-equivariant minimal surface and induces a cyclic harmonic metric of $(\mathcal{E}, \varphi)$.*

Conversely, every cyclic $\mathrm{SL}_n\mathbb{C}$-Higgs bundle over a Riemann surface (Σ, j) with cyclic harmonic metric h and nowhere vanishing $\mathrm{pr}_{\mathfrak{g}_\alpha}(\varphi)$ induces a α-cyclic surface (ρ, f) from Σ.

Infinitesimal Rigidity

Variation of Cyclic Surfaces

Definition

A smooth variation of equivariant α-surface is a smooth map $F\colon \widetilde{\Sigma} \times (-\varepsilon, \varepsilon) \to \mathbb{X}$ such that $f_t(\bullet) := F(\bullet, t)$ is a ρ_t-equivariant α-cyclic surface for every t, where $(\rho_t\colon \pi_1(\Sigma) \to \mathrm{SL}_n\mathbb{C})_{t\in(-\varepsilon,\varepsilon)}$ is a smooth path of representations.

Variation of Cyclic Surfaces

Definition

A smooth variation of equivariant α-surface is a smooth map $F\colon \widetilde{\Sigma} \times (-\varepsilon, \varepsilon) \to \mathbb{X}$ such that $f_t(\bullet) := F(\bullet, t)$ is a ρ_t-equivariant α-cyclic surface for every t, where $(\rho_t\colon \pi_1(\Sigma) \to \mathrm{SL}_n\mathbb{C})_{t\in(-\varepsilon,\varepsilon)}$ is a smooth path of representations.

We consider the first order deformation (**Jacobi field**) tangent to F, i.e. $\dot{f}_0 := \mathrm{d}F(\partial_t)|_{t=0}$, which can be viewed as a smooth section of $f_0^*\mathrm{T}\mathbb{X} \to \widetilde{\Sigma}$. Let f_0 induce the cyclic $\mathrm{SL}_n\mathbb{C}$-Higgs bundle $(\mathcal{E}, \varphi)$ over the Riemann surface (Σ, j) with cyclic harmonic metric h.

Variation of Cyclic Surfaces

Definition

A smooth variation of equivariant α-surface is a smooth map $F\colon \widetilde{\Sigma} \times (-\varepsilon, \varepsilon) \to \mathbb{X}$ such that $f_t(\bullet) := F(\bullet, t)$ is a ρ_t-equivariant α-cyclic surface for every t, where $(\rho_t\colon \pi_1(\Sigma) \to \mathrm{SL}_n\mathbb{C})_{t\in(-\varepsilon,\varepsilon)}$ is a smooth path of representations.

We consider the first order deformation (**Jacobi field**) tangent to F, i.e. $\dot{f}_0 := \mathrm{d}F(\partial_t)|_{t=0}$, which can be viewed as a smooth section of $f_0^*\mathrm{T}\mathbb{X} \to \widetilde{\Sigma}$. Let f_0 induce the cyclic $\mathrm{SL}_n\mathbb{C}$-Higgs bundle $(\mathcal{E}, \varphi)$ over the Riemann surface (Σ, j) with cyclic harmonic metric h.

When $\dot{\rho}_0 = 0$, then $\dot{f}_0$ descends to a smooth section ξ_F of $\pi_* f_0^*\mathrm{T}\mathbb{X} \to \Sigma$.

Variation of Cyclic Surfaces

Definition

A smooth variation of equivariant α-surface is a smooth map $F\colon \widetilde{\Sigma} \times (-\varepsilon, \varepsilon) \to \mathbb{X}$ such that $f_t(\bullet) := F(\bullet, t)$ is a ρ_t-equivariant α-cyclic surface for every t, where $(\rho_t\colon \pi_1(\Sigma) \to \mathrm{SL}_n\mathbb{C})_{t\in(-\varepsilon,\varepsilon)}$ is a smooth path of representations.

We consider the first order deformation (**Jacobi field**) tangent to F, i.e. $\dot{f}_0 := \mathrm{d}F(\partial_t)|_{t=0}$, which can be viewed as a smooth section of $f_0^*\mathrm{T}\mathbb{X} \to \widetilde{\Sigma}$. Let f_0 induce the cyclic $\mathrm{SL}_n\mathbb{C}$-Higgs bundle $(\mathcal{E}, \varphi)$ over the Riemann surface (Σ, j) with cyclic harmonic metric h.

When $\dot{\rho}_0 = 0$, then $\dot{f}_0$ descends to a smooth section ξ_F of $\pi_* f_0^*\mathrm{T}\mathbb{X} \to \Sigma$.

Note that the metric h on $\mathcal{E}$ induces a metric on $\mathrm{End}(\mathcal{E})$ and we have $\pi_* f_0^*\mathrm{T}\mathbb{X} \subset \mathrm{End}(\mathcal{E})$.

Admissible Variation

Let g be a complete conformal metric on $X = (\Sigma, \mathrm{j})$ and $B(x, r; g)$ denote the geodesic ball centered at $x \in \Sigma$ with radius r.

Admissible Variation

Let g be a complete conformal metric on $X = (\Sigma, \mathrm{j})$ and $B(x, r; g)$ denote the geodesic ball centered at $x \in \Sigma$ with radius r.

Definition

We call F an **admissible variation** with respect to g if there exists a point $x \in \Sigma$ such that

$$\liminf_{r \to +\infty} \frac{1}{r} \int_{B(x,r;g)} |\xi_F|_h^2 \mathrm{d}\,\mathrm{vol}_g = 0$$

and

$$\liminf_{r \to +\infty} \frac{1}{r} \int_{B(x,r;g)} |\xi_F|_h^2 \cdot |\varphi|_{h,g} \mathrm{d}\,\mathrm{vol}_g = 0.$$

Remark

- *When* $\Phi := |\varphi|_h \otimes |\varphi^{*_h}|_h$ *defines a complete conformal metric on* X, F *is admissible with respect to* Φ *is equivalent to say that there exists a point* $x \in \Sigma$ *such that*

$$\liminf_{r \to +\infty} \frac{1}{r} \int_{B(x,r;\Phi)} |\xi_F|_h^2 \mathrm{d}\,\mathrm{vol}_\Phi = 0.$$

Remark

- *When* $\Phi := |\varphi|_h \otimes |\varphi^{*_h}|_h$ *defines a complete conformal metric on* X, F *is admissible with respect to* Φ *is equivalent to say that there exists a point* $x \in \Sigma$ *such that*

$$\liminf_{r\to+\infty} \frac{1}{r} \int_{B(x,r;\Phi)} |\xi_F|_h^2 \mathrm{d}\,\mathrm{vol}_\Phi = 0.$$

- *In particular, if both* $|\xi_F|_h^2$ *and* $|\xi_F|_h^2 \cdot |\varphi|_{h,g}$ *are* L^1*-functions with respect to* g, *then* F *is admissible.*

Remark

- *When* $\Phi := |\varphi|_h \otimes |\varphi^{*_h}|_h$ *defines a complete conformal metric on* X, F *is admissible with respect to* Φ *is equivalent to say that there exists a point* $x \in \Sigma$ *such that*

$$\liminf_{r\to+\infty} \frac{1}{r} \int_{B(x,r;\Phi)} |\xi_F|_h^2 \mathrm{d}\,\mathrm{vol}_\Phi = 0.$$

- *In particular, if both* $|\xi_F|_h^2$ *and* $|\xi_F|_h^2 \cdot |\varphi|_{h,g}$ *are* L^1*-functions with respect to* g, *then* F *is admissible.*
- *When* Σ *is closed, every variation is admissible.*

Theorem (Li–Z., 2026+)

Given a smooth variation of equivariant α-cyclic surface (ρ_t, f_t) from Σ with $\dot{\rho}_0 = 0$.

Theorem (Li–Z., 2026+)

Given a smooth variation of equivariant α-cyclic surface (ρ_t, f_t) from Σ with $\dot{\rho}_0 = 0$. If

- *(ρ_t, f_t) is admissible with respect to some complete comformal metric on X;*

Theorem (Li–Z., 2026+)

Given a smooth variation of equivariant α-cyclic surface (ρ_t, f_t) from Σ with $\dot{\rho}_0 = 0$. If

- *(ρ_t, f_t) is admissible with respect to some complete comformal metric on X;*
- *$\mathrm{im}(f_0)$ does not lie in some $S(\mathrm{GL}(W_1) \times \mathrm{GL}(W_2))$-orbit of $\mathbb{X}$ for a proper orthogonal decomposition $\mathbb{C}^n = W_1 \oplus W_2$,*

Theorem (Li–Z., 2026+)

Given a smooth variation of equivariant α-cyclic surface (ρ_t, f_t) from Σ with $\dot{\rho}_0 = 0$. If

- *(ρ_t, f_t) is admissible with respect to some complete comformal metric on X;*
- *$\mathrm{im}(f_0)$ does not lie in some $S(\mathrm{GL}(W_1) \times \mathrm{GL}(W_2))$-orbit of $\mathbb{X}$ for a proper orthogonal decomposition $\mathbb{C}^n = W_1 \oplus W_2$,*

then there exists a smooth path

$$(\psi_t) \subset \mathrm{Diff}^0_+(S)$$

such that

$$(\rho_t, f_t \circ \psi_t)$$

*is a smooth variation of equivariant α-cyclic surfaces with **zero** Jacobi field, i.e.*

$$\left.\frac{\mathrm{d}(f_t \circ \psi_t)}{\mathrm{d}t}\right|_{t=0} = 0.$$

Corollary (Li–Z., 2026+)

*Given a smooth variation of equivariant α-cyclic surface (ρ_t, f_t) from a **closed surface** Σ with $\dot{\rho}_0 = 0$.*

Corollary (Li–Z., 2026+)

Given a smooth variation of equivariant α-cyclic surface (ρ_t, f_t) from a ***closed surface*** *Σ with $\dot{\rho}_0 = 0$. If the Higgs bundle $(\mathcal{E}, \varphi)$ induced by f_0 is stable,*

Corollary (Li–Z., 2026+)

*Given a smooth variation of equivariant α-cyclic surface (ρ_t, f_t) from a **closed surface** Σ with $\dot{\rho}_0 = 0$. If the Higgs bundle $(\mathcal{E}, \varphi)$ induced by f_0 is stable, then there exists a smooth path*

$$(\psi_t) \subset \mathrm{Diff}^0_+(S)$$

such that

$$(\rho_t, f_t \circ \psi_t)$$

*is a smooth variation of equivariant α-cyclic surfaces with **zero** Jacobi field, i.e.*

$$\left.\frac{\mathrm{d}(f_t \circ \psi_t)}{\mathrm{d}t}\right|_{t=0} = 0.$$

Corollary (Li–Z., 2026+)

*Given a smooth variation of equivariant α-cyclic surface (ρ_t, f_t) from a **closed surface** Σ with $\dot{\rho}_0 = 0$. If the Higgs bundle $(\mathcal{E}, \varphi)$ induced by f_0 is stable, then there exists a smooth path*

$$(\psi_t) \subset \mathrm{Diff}^0_+(S)$$

such that

$$(\rho_t, f_t \circ \psi_t)$$

*is a smooth variation of equivariant α-cyclic surfaces with **zero** Jacobi field, i.e.*

$$\left.\frac{\mathrm{d}(f_t \circ \psi_t)}{\mathrm{d}t}\right|_{t=0} = 0.$$

Remark

- *The whole set-up and our theorem could be generalized to an arbitrary connected semisimple complex Lie group.*

Corollary (Li–Z., 2026+)

*Given a smooth variation of equivariant α-cyclic surface (ρ_t, f_t) from a **closed surface** Σ with $\dot{\rho}_0 = 0$. If the Higgs bundle $(\mathcal{E}, \varphi)$ induced by f_0 is stable, then there exists a smooth path*

$$(\psi_t) \subset \mathrm{Diff}^0_+(S)$$

such that

$$(\rho_t, f_t \circ \psi_t)$$

*is a smooth variation of equivariant α-cyclic surfaces with **zero** Jacobi field, i.e.*

$$\left.\frac{\mathrm{d}(f_t \circ \psi_t)}{\mathrm{d}t}\right|_{t=0} = 0.$$

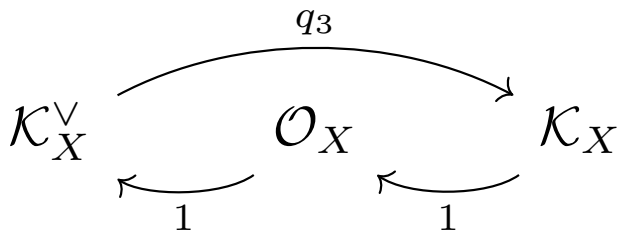
Remark

- *The whole set-up and our theorem could be generalized to an arbitrary connected semisimple complex Lie group.*
- *When Σ is closed, this has been proven by* [Collier–Toulisse–Wentworth, 2025+] *with a very different approach (constructing a "joint moduli space").*

Previous Results

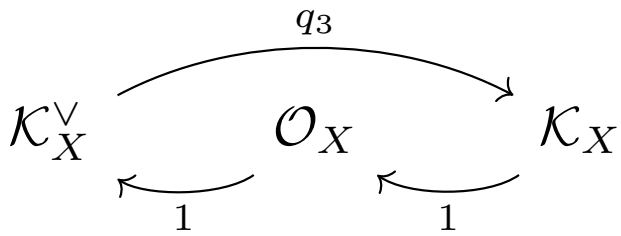
Previous Results

- Hitchin representations for $\mathrm{SL}_3\mathbb{R}$

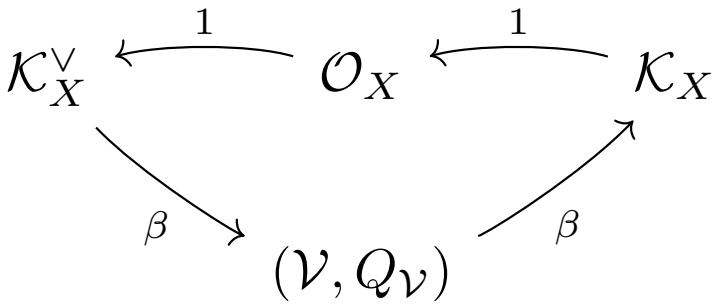


Previous Results

- Hitchin representations for $\mathrm{SL}_3\mathbb{R}$

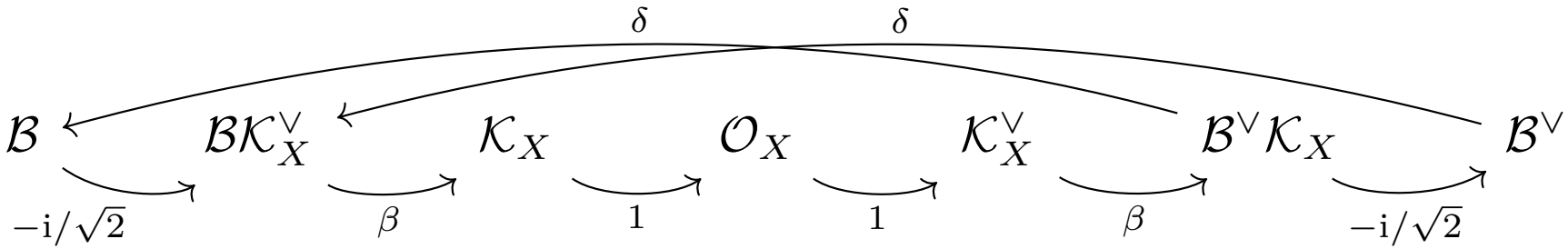


- Maixmal representations for $\mathrm{SO}^0_{2,n+1}$



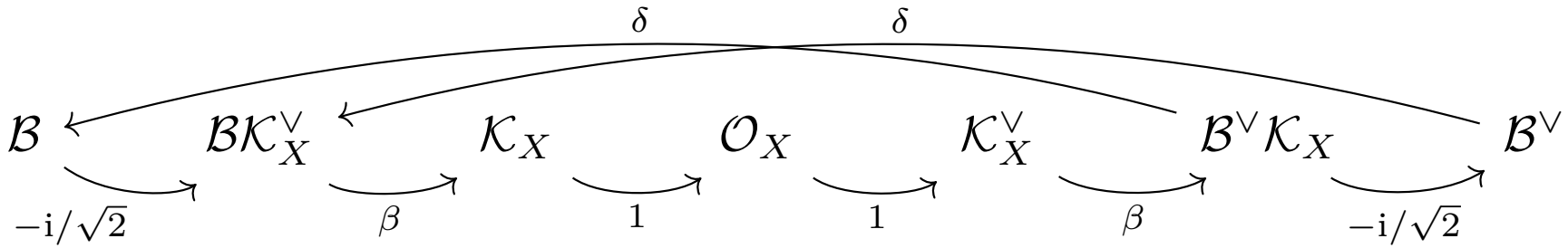
Previous Results

- Alternating J-holomorphic curves in $\mathbb{H}^{4,2}$

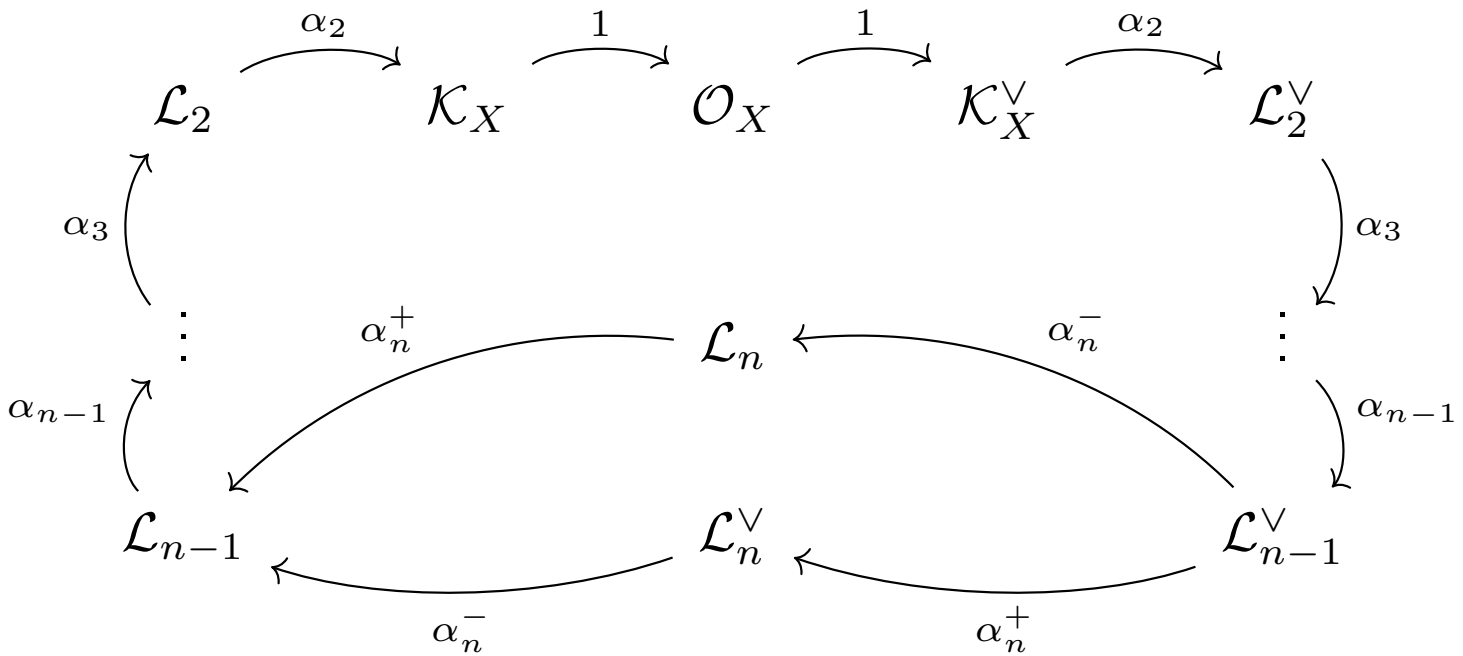


Previous Results

- Alternating J-holomorphic curves in $\mathbb{H}^{4,2}$

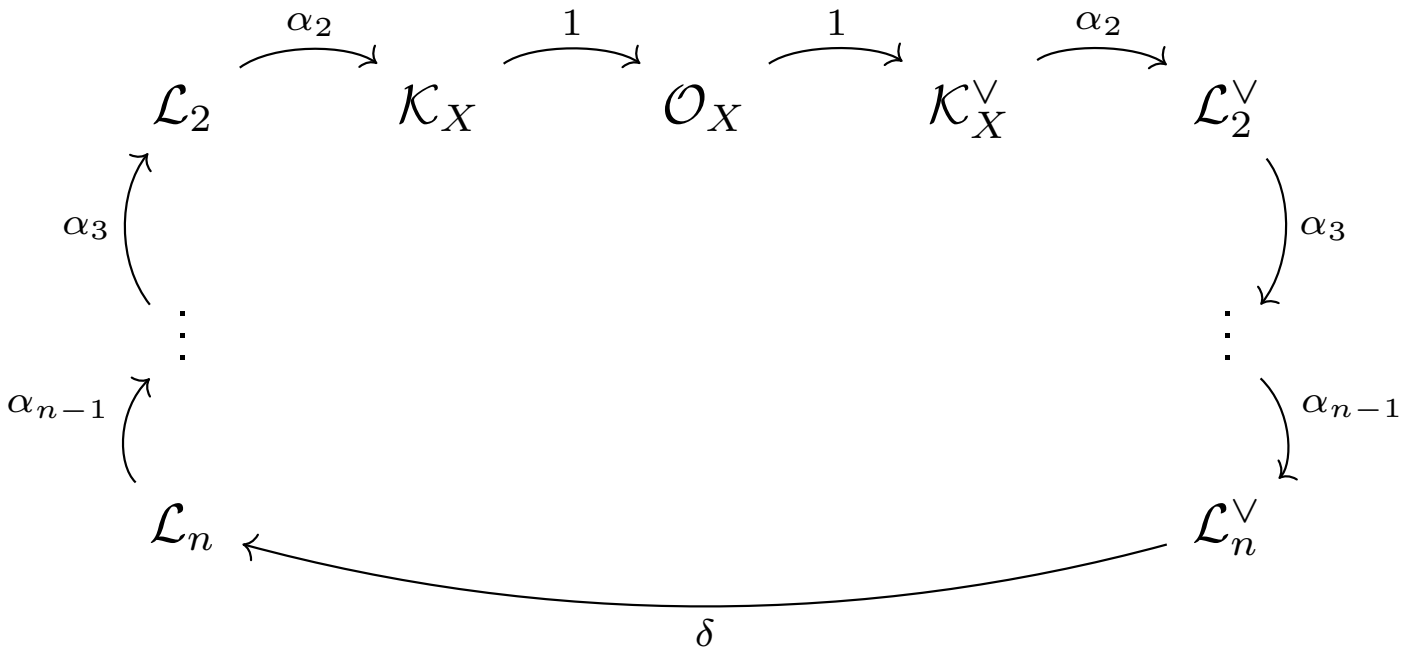


- Alternating curves in $\mathbb{H}^{n,n}$ or $\mathbb{H}^{n+1,n-1}$



Previous Results

- Alternating curves in para-complex hyperbolic space



Sketch of Proof

Sketch of Proof

(1) Interpret cyclic surfaces as solutions of a specific Pffafian system (Labourie's approach)

Sketch of Proof

(1) Interpret cyclic surfaces as solutions of a specific Pffafian system (Labourie's approach) & $\mathcal{J}$-holomorphic curves;

Sketch of Proof

(1) Interpret cyclic surfaces as solutions of a specific Pffafian system (Labourie's approach) & $\mathcal{J}$-holomorphic curves;

(2) Prove that the normal projection $\xi_F^{\perp}$ of the Jacobi field is an infinitesimal automorphism of $(\mathcal{E}, \varphi)$, i.e. $\xi_F^{\perp}$ is holomorphic and $[\xi_F^{\perp} \wedge \varphi] = 0$;

Sketch of Proof

(1) Interpret cyclic surfaces as solutions of a specific Pffafian system (Labourie's approach) & $\mathcal{J}$-holomorphic curves;

(2) Prove that the normal projection $\xi_F^{\perp}$ of the Jacobi field is an infinitesimal automorphism of $(\mathcal{E}, \varphi)$, i.e. $\xi_F^{\perp}$ is holomorphic and $[\xi_F^{\perp} \wedge \varphi] = 0$;

(3) Show the admissible infinitesimal automorphism of $(\mathcal{E}, \varphi)$ must be 0.

Thank you!