Anosov Property of Some Specific $\mathrm{SO}_0(2,3)$-Higgs bundles
(in progress)

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Classical Teichmüller Theory

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$$\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R}).$$

Moreover, $\mathcal{T}(S)$ is one of the two connected components which consist entirely of discrete and faithful representations. The other one is $\mathcal{T}(\overline{S})$, where $\overline{S}$ denotes the surface S with the opposite orientation.

Anosov Property of Fuchsian Representation

We fix a base point $x_0 = (0,1) \in \mathbb{H}^2 \cong \widetilde{S}$ on the universal cover of S. For a Fuchsian representation $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$, the well-known Milnor–Švarc lemma tells us the orbit map

$$\begin{aligned} \tau_\rho\colon \pi_1(S) &\to \mathbb{H}^2 \cong \mathrm{PSL}(2,\mathbb{R})/\mathrm{PSO}(2) \\ \gamma &\mapsto \rho(\gamma)(x_0) \end{aligned}$$

is a quasi-isometry, which shows that there exist constants $D, L > 0$ such that

$$\ln \frac{\sigma_1(\rho(\gamma))}{\sigma_2(\rho(\gamma))} = d_{\mathbb{H}^2}(x_0, \rho(\gamma)(x_0)) \geqslant D \cdot d(1,\gamma) - L,$$

where $d(1,\gamma)$ means the word length of γ in $\pi_1(S)$ and σ_i denotes the i-th singular value.

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A suitable way to generalize is using the inequality related to the singular values.

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The original definition is highly dynamical and we will use an interpreted definition (which is proven equivalent to the original one by Kapovich–Leeb–Porti and Bochi–Potrie–Sambarino).

Also, we will assume G is a semi-simple Lie subgroup of $\mathrm{SL}(n, \mathbb{R})$ here to avoid involving a Lie-theoretic description.

Definition

$\rho\colon \pi_1(S) \to G$ is P_k-**Anosov** if there exist constants $D, L > 0$ such that

$$\ln \frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))} \geqslant D \cdot d(1,\gamma) - L, \forall \gamma \in \pi_1(S),$$

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Anosov $\Longrightarrow$ **discrete** + **faithful**.

Anosov property plays an important role in higher Teichmüller theory.

Higher Teichmüller Spaces

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- (Hitchin, 1992; Labourie, 2006; Fock–Goncharov,2006) Hitchin components $\mathcal{T}_{Hit}(S, G)$ for **real split** G, such as $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{SO}_0(n, n+1)$, $\mathrm{Sp}(2n, \mathbb{R})$ and so on;

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When $G = \mathrm{SL}(2, \mathbb{R})$, the above components coincide with the classical Teichmüller space.

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We would like to find some new Anosov representations.

Non-Abelian Hodge Correspondence

Higgs bundles

The Higgs bundle is a useful tool to study the higher Teichmüller space. It is usually used to give a parametrization of the higher Teichmüller space. We fix a complex structure on S such that it becomes a Riemann surface X. Let $\mathcal{K}_X$ be its canonical line bundle.

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Definition

A **($\mathrm{GL}(n,\mathbb{C})$-)Higgs bundle** over X is a pair $(\mathcal{E},\Phi)$ consisting of the following data:

- a holomorphic vector bundle $\mathcal{E}$ over X with $\mathrm{rank}(\mathcal{E})=n$;
- a holomorphic section $\Phi\in\mathrm{H}^0(X,\mathrm{End}(\mathcal{E})\otimes\mathcal{K}_X)$ called **Higgs field**.

The non-Abelian Hodge correspondence exhibit a homeomorphism between the following moduli spaces:

$$\{\textbf{reductive} \text{ representation } \rho\colon \pi_1(S) \to \mathrm{GL}(n,\mathbb{C})\}/\mathrm{GL}(n,\mathbb{C})$$

associated bundle ↓ ↑ holonomy

$$\{\textbf{reductive} \text{ flat vector bundle } (E \to S, \nabla) \text{ with } \operatorname{rank}(E) = n\}/\mathcal{G}$$

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harmonic metric

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$$\{\textbf{polystable} \text{ Higgs bundle}(\mathcal{E} \to X, \Phi) \text{ with } \operatorname{rank}(\mathcal{E}) = n, \deg(\mathcal{E}) = 0\}/\mathcal{G}$$

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If we equip additional structure on these objects, we can get the non-Abelian Hodge correspondence for general reductive Lie group G.

Hitchin–Kobayashi Correspondence (from Higgs to flat)

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Theorem (Hitchin–Simpson)

If $(\mathcal{E}, \Phi)$ is a polystable Higgs bundle with $\deg(\mathcal{E}) = 0$, then there exists an Hermitian metric h on $\mathcal{E}$ such that

$$F(\nabla^h) + [\Phi, \Phi^{*_h}] = 0, \tag{1}$$

where ∇^h is the Chern connection of the metric h, $F(\nabla^h)$ denotes its curvature form and Φ^{*_h} is the adjoint of Φ with respect to h. Moreover, if $(\mathcal{E}, \Phi)$ is stable, then such h is unique up to a constant scalar.

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If h solves (1), then

$$\nabla^h + \Phi + \Phi^{*_h}$$

gives a flat connection.

$$\text{Higgs bundle} \xrightarrow[]{\textbf{Hitchin's self-dual equation}} \text{Anosov property?}$$

Example: Hitchin Component in Higgs Bundle Viewpoint

Let us fix a square root $\mathcal{K}_X^{1/2}$ of $\mathcal{K}_X$, then the Hitchin component for $\mathrm{SL}(n,\mathbb{R})$ consisting of entirely the Higgs bundles of the following form:

$$\mathcal{E} = \mathcal{K}_X^{(n-1)/2} \oplus \mathcal{K}_X^{(n-3)/2} \oplus \cdots \oplus \mathcal{K}_X^{(1-n)/2},$$

$$\Phi = \begin{pmatrix} 0 & q_2 & q_3 & q_4 & \cdots & q_n \\ 1 & 0 & q_2 & q_3 & \ddots & q_{n-1} \\ & 1 & 0 & q_2 & \ddots & \vdots \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & 1 & 0 & q_2 \\ & & & & 1 & 0 \end{pmatrix},$$

where $1\colon \mathcal{K}_X^{(n-1)/2-i} \to \mathcal{K}_X^{(n-1)/2-(i+1)} \otimes \mathcal{K}_X$ is the natural isomorphism and $q_i \in \mathrm{H}^0(X, \mathcal{K}_X^i)$.

It corresponds to the component containing the embedding of Fuchsian representations through the unique irreducible $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R})$.

From $\mathrm{SO}_0(2,3)$-Higgs Bundle to Anosov Representation

Special $\mathrm{SO}_0(2,3)$-Higgs Bundles

Below we consider the Higgs bundle whose underlying bundle is

$$\mathcal{E} = \mathcal{L}_{-2} \oplus \mathcal{L}_{-1} \oplus \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2,$$

where $\mathcal{L}_i$ are line bundles with $\mathcal{L}_i \cong \mathcal{L}_{-i}^\vee$. Note that there is a natural pairing on $\mathcal{E}$ defined by

$$Q = \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & 1 & & \\ & -1 & & & \\ 1 & & & & \end{pmatrix}$$

Suppose that the Higgs field Φ projects to 0 in $\mathrm{H}^0(X, \mathrm{Hom}(\mathcal{L}_i, \mathcal{L}_j) \otimes \mathcal{K}_X)$ for any i, j have the same parity and Φ is compatible with Q, then polystable $(\mathcal{E}, \Phi)$ gives an $\mathrm{SO}_0(2,3)$-representation.

In addition, if $(\mathcal{E}, \Phi)$ comes from a variation of Hodge structure, then we have

$$(\mathcal{E}, \Phi) = \mathcal{L}_{-2} \xrightarrow{\alpha} \mathcal{L}_{-1} \xrightarrow{\beta} \mathcal{L}_0 \xrightarrow{\beta} \mathcal{L}_1 \xrightarrow{\alpha} \mathcal{L}_2 .$$

Such Higgs bundle is maximal if and only if β is an isomorphism, and maximal representations are known to be Anosov. From a different starting point, S. Filip considered the Higgs bundle of the same form, but instead α is an isomorphism.

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Theorem (Filip, 2021)

A stable $\mathrm{SO}_0(2,3)$-Higgs bundle of the form

$$\mathcal{L}_{-2} \xrightarrow{\alpha} \mathcal{L}_{-1} \xrightarrow{\beta} \mathcal{L}_0 \xrightarrow{\beta} \mathcal{L}_1 \xrightarrow{\alpha} \mathcal{L}_2$$

with α is an isomorphism gives a P_2-Anosov representation.

Filip proved this theorem by an **analytic method** and used this to show a equality connecting the Lyapunov exponents and the Chern classes conjectured by Eskin–Kontsevich–Möller–Zorich, which was inspired by a conjecture of Fei Yu, i.e., the flat bundle associated with the Higgs bundle satisfying that

$$2\lambda_1 = \frac{\deg \mathcal{L}_2}{\chi(X)},$$

where $\lambda_1 \geqslant 0$ is the first Lyapunov exponent and $\chi(X)$ is the Euler characteristic of X.

Inspired by his method and with some simplification, we extend his results and discover the Anosov property of a general family of $\mathrm{SO}_0(2,3)$-Higgs bundles.

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Theorem (Z.)

A stable $\mathrm{SO}_0(2,3)$-Higgs bundle of the form

$$\begin{tikzcd} \mathcal{L}_{-2} \arrow[r, "\alpha"'] & \mathcal{L}_{-1} \arrow[r, "\beta"'] & \mathcal{L}_{0} \arrow[r, "\beta"'] & \mathcal{L}_{1} \arrow[r, "\alpha"'] \arrow[lll, bend right, "\gamma"'] & \mathcal{L}_{2} \arrow[lll, bend right, "\gamma"'] \end{tikzcd}$$

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Sketch of Proof

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When lifting to the universal cover $\mathbb{H}^2$, suppose

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is the global flat section with a suitable initial vector $v \in (\mathcal{E})_{x_0}$. The section v decomposes as $\sum_{i=-2}^{2} v_i$ with $v_i\colon \mathbb{H}^2 \to \mathcal{L}_i$.

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Let

$$f_v\colon \mathbb{H}^2 \to \mathbb{R}$$
$$x \mapsto |v_2(x)|_h^2.$$

A Lie-theoretic analysis shows that if there exist constants $C_1, C_2, \varepsilon > 0$ such that

$$f_v(x) \geqslant C_1 \cdot \exp(\varepsilon \cdot d_{\mathbb{H}^2}(x, x_0)) - C_2,$$

then the corresponding ρ is P_2-Anosov.

Thank you!