Which spheres are Lie groups?

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Abstract. In this article, we will prove that a Lie group is a sphere if and only if it has dimension 0,1 or 3 and construct the covering homomorphism from $SU(2)$ to $SO(3)$.

1. A brief introduction to Lie group and Lie algebra

We denote the standard Euclidean space and the sphere with dimension n by $\mathbb{R}^n$ and S^n respectively.

Definition 1.1. *A Lie group is a smooth manifold G with a group structure which makes*

$$G \times G \to G$$
$$(g, h) \mapsto gh^{-1}$$

smooth.

Definition 1.2. *For a Lie group G and an arbitrary $g \in G$, the map*

$$l_g : G \to G$$
$$h \mapsto gh$$

is called the left translation by g. It is a diffeomorphism. Similarly, we could define the right translation by g, i.e. $r_g := [h \mapsto hg]$.

When $l_g = r_g$ for all $g \in G$, G is called an abelian Lie group.

Example 1.1. *$\mathbb{R}^n$ is an abelian Lie group. For $x, y \in \mathbb{R}^n$, $l_x y = r_x y = x+y$.*

Example 1.2. *S^1 is an abelian Lie group. When we regard S^1 as $\mathbb{R}/2\pi\mathbb{Z}$, $l_{x+2\pi\mathbb{Z}}(y + 2\pi\mathbb{Z}) = x + y + 2\pi\mathbb{Z}$.*

Example 1.3. *S^0 is a disconnected Lie group.*

For two smooth manifolds M, N, a point $p \in M$ and a smooth map $f : M \to N$, let T_pM, TM, $f_* : TM \to TN$ denote the tangent space of M at p, the tangent bundle of M, the differential of f respectively.

Definition 1.3. *For a Lie group G, if a section $s : G \to TG$, i.e. s is a continuous map and $s(g) \in T_gG$ for any $g \in G$, satisfies*

$$s \circ l_g = (l_g)_* \circ s,$$

then s is called a left invariant vector field.

Remark 1.1. *s is automatically smooth. Its proof could be found in* [1].

For a Lie group G with the identity e, denote the collection of its left invariant vector fields by $\mathfrak{g}$. $\mathfrak{g}$ is isomorphism to T_eG as linear space. For instance, $[X \mapsto X(e)]$ has the linear inverse $[v \mapsto [g \mapsto (l_g)_* v]]$.

Similarly we could define left invariant form, right invariant vector field and right invariant form. A vector field or a form is called bi-invariant if it is both left and right invariant.

Definition 1.4. *For a field F, a Lie algebra over F is a linear space V equipped with a bilinear map $[-,-] : V \times V \to V$ which satisfies*

(a)(anti-commutativity)$[x,x] = 0, \forall x \in V$;

(b)(Jacobi's identity)$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0, \forall x,y,z \in V$. $[-.-]$ is called the Lie bracket.

A Lie algebra is called abelian if $[-,-] \equiv 0$. Now let $[-,-]$ be the Lie bracket defined on the smooth vector fields of a smooth manifold, by the following proposition we could claim $\mathfrak{g}$ is a Lie algebra over $\mathbb{R}$ which has the same dimension with G.

Proposition 1.1. *For two arbitrary $X, Y \in \mathfrak{g}$, $[X,Y] \in \mathfrak{g}$.*

Thus $\mathfrak{g}$ is called the Lie algebra of G.

Definition 1.5. *An abstract group homomorphism between two Lie groups G, H is called a (Lie group) homomorphism when it is smooth. A Lie group homomorphism is called a (Lie group) isomorphism if it is a diffeomorphism.*

Example 1.4. *The natural quotient map from $\mathbb{R}$ to $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ is a homomorphism.*

Definition 1.6. *For two Lie algebras $\mathfrak{g}$, $\mathfrak{h}$, a linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ is called a (Lie algrbra) homomorphism if it preserves the Lie bracket, i.e. $\varphi([x,y]) = [\varphi(x), \varphi(y)]$. A Lie algebra homomorphism is called a (Lie algebra) isomorphism if it is invertible.*

The Lie group homomorphism will induce a Lie algebra homomorphism.

Theorem 1.1. *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively, and let $\varphi : G \to H$ be a homomorphism. Then $\varphi_* : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.*

Conversely, Lie algebra homomorphism sometimes raise to a Lie group homomorphism.

Theorem 1.2. *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively and with G simply connected. Let $\psi : \mathfrak{g} \to \mathfrak{h}$ be a homomorphism. Then there exists a unique homomorphism $\varphi : G \to H$ such that $\varphi_* = \psi\cdot$*

Another object we need is the exponential map.

Definition 1.7. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, for any $v \in T_eG$, there exists a unique Lie group homomorphism $\gamma_v : \mathbb{R} \to G$ such that $\gamma_v'(0) = v$, γ_v is called the one parameter subgoup of G associated with v and the map*

$$\begin{aligned} \exp : \mathfrak{g} &\to G \\ X &\mapsto \gamma_{X(e)}(1) \end{aligned}$$

is called the exponential map.

In the end, we introduce the covering group, let G be a connected Lie group.

Theorem 1.3. *If $\tilde{G}$ is a connected manifold and $\varphi : \tilde{G} \to G$ is a covering, then $\tilde{G}$ has a unique structure of a Lie group such that φ is a homomorphism.*

Theorem 1.4. *A Lie group homomorphism is a covering if and only if its differential is an isomorphism.*

The proofs of theorems above could be found in [1] and [5].

2. Bi-invariant metric and bi-invariant form

We first explain what is a bi-invariant metric.

Definition 2.1. *A Riemannian metric on a Lie group G is left invariant if*

$$\langle X, Y\rangle_h = \langle (l_g)_*X, (l_g)_*Y\rangle_{gh}, \forall g, h \in G, \forall X, Y \in T_hG.$$

Similarly we could define right invariant metric. If a Riemannian metric is both left invariant and right invariant, it is said to be bi-invariant.

An important question is which Lie group could admit a bi-invariant metric. In fact, we have the following theorem.

Theorem 2.1. *A compact connected Lie group G admits a bi-invariant metric.*

Proof. Let $n := \dim G$. By the left translation, we could define a left invariant positive $n-$form ω and a left invariant metric $\langle -, -\rangle$ on G. Then

$$(X, Y)_g = \int_G \langle (r_h)_*(X), (r_h)_*(Y)\rangle_{gh}\omega(h)$$

defines a bi-invariant metric $(-, -)$. □

Its converse is also true in some sense, John Milnor proved the following theorem in [2].

Theorem 2.2. *The connected Lie group G admits a bi-invariant metric if and only if it is isomorphic to the cartesian product of a compact group and an additive vector group.*

The bi-invariant metric restricted on $\mathfrak{g}$ has the following relation.

Theorem 2.3. *For a bi-invariant metric $\langle -, -\rangle$ and $X, Y, Z \in \mathfrak{g}$,*

$$\langle [X, Y], Z\rangle = \langle X, [Y, Z]\rangle.$$

Proof. Let $Ad_g = (l_g r_{g^{-1}})_*$ be the adjoint action on $\mathfrak{g}$ and Y_t be the flow induced by Y, i.e. $Y_t(g) = \gamma_g(t)$, where γ_g is the integral curve of Y in a neighborhood of g and $\gamma_g(0) = g$. Note that $l_g \circ Y_t$ and $Y_t \circ l_g$ are both the solution of the first order ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} f_t(p) = Y(f_t)(p), f_0 = l_g,$$

so $l_g \circ Y_t = Y_t \circ l_g$. Thus $Y_t(g) = gY_t(e) = r_{Y_t(e)}(g)$, which means that $(Y_t)_* = (r_{Y_t(e)})_*$. So through $Ad_g(X) = (r_{g^{-1}})_*(X \circ l_g)$ we get

$$[X, Y] = \lim_{t\to 0} \frac{Ad_{(Y_t(e))^{-1}}(X) - X}{t}.$$

Then by differentiating $\langle X, Z\rangle = \langle (r_{Y_t(e)})_*X, (r_{Y_t(e)})_*Y\rangle$ and set $t = 0$ we prove

$$0 = \langle [X, Y], Z\rangle + \langle X, [Z, Y]\rangle.$$

Thus the result follows. □

Corollary 2.1. *Let $\langle -, -\rangle$ be a bi-invariant metric on G, then*

$$t(X, Y, Z) := \langle [X, Y], Z\rangle$$

is a bi-invariant $3-$form on G.

Proof. Let $\{v_i\}$ be a basis of $\mathfrak{g}$, then for a smooth vector field X on G, there exists a collection of smooth functions $\{X^i\}$ such that $X = X^i v_i$, by $\langle -, - \rangle$ is a bi-invariant 2−form and the theorem above we conclude that the corollary follows. □

Now we would like to prove that bi-invariant form defines a cohomology class, i.e. bi-invariant form is closed. We first prove a lemma.

Lemma 2.1. *Let ψ be the map $g \mapsto g^{-1}$ on a Lie group G. A form ω is left invariant if and only if $\psi^*\omega$ is right invariant. And the value of a $k-$form ω at the identity e satisfies that $\psi^*\omega(e) = (-1)^k\omega(e)$*

Proof. By $\psi \circ r_g = l_{g^{-1}} \circ \psi$ we could get

$$(r_g)^*\psi^*\omega = \psi^*(l_{g^{-1}})^*\omega,$$

thus the first result follows. For the second one, it suffices to prove it when $k = 1$, i.e. for a $X \in T_eG$, $\psi_*X = -X$. X is just the tangent vector of the curve $[t \mapsto \exp tX]$, so ψ_*X is the tangent vector of the curve $[t \mapsto \exp -tX]$, which is just $-X$. □

Theorem 2.4. *A bi-invariant form is closed.*

Proof. Let ψ be the map $[g \mapsto g^{-1}]$ and ω is a bi-invariant $k-$form, we have shown that $\psi^*\omega(e) = (-1)^k\omega(e)$. Because ω and $\psi^*\omega$ are both left invariant, $\psi^*\omega = (-1)^k\omega$. Similarly, $\psi^*\mathrm{d}\omega = (-1)^{k+1}\mathrm{d}\omega$ since $\mathrm{d}\omega$ is a bi-invariant $k+1-$form. But

$$\psi^*\mathrm{d}\omega = \mathrm{d}\psi^*\omega = (-1)^k\mathrm{d}\omega,$$

that implies $\mathrm{d}\omega = 0$. □

As a corollary, we could claim that abelian Lie group has abelian Lie algebra.

Corollary 2.2. *Let G be an abelian Lie group with abelian Lie algebra $\mathfrak{g}$, then $\mathfrak{g}$ is also abelian.*

Proof. For any $X, Y \in \mathfrak{g}$ and a nonzero left invariant 1−form ω,

$$\mathrm{d}\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]) = -\omega([X, Y]).$$

Since G is abelian, ω is bi-invariant and $\mathrm{d}\omega = 0$, which means $[X, Y] = 0$. □

Form this we can conclude the following corollary.

Corollary 2.3. *When $n \geqslant 1$, S^n could be an abelian Lie group if and only if $n = 1$.*

Proof. If S^n is an abelian Lie group, it will have abelian Lie algebra, then by Theorem 1.2 there is a Lie group homomorphism from $\mathbb{R}^n$ to S^n, and by Theorem 1.4 we conclude that $\mathbb{R}^n$ is the universal cover of S^n. That is true if and only if $n = 1$. On the other hand, we have shown that S^1 is an abelian Lie group. □

3. The third de Rham cohomology group

For the nonabelian case, we would prove that a nonabelian connected compact Lie group's third de Rham cohomology group is nontrivial. We first introduce the result that the de Rham cohomology group of a compact

Lie group, (or much stronger, a homogeneous space) could be reduced to the invariant form version without proof.

Proposition 3.1. *Let G be a connected compact Lie group. The k^{th} de Rham cohomology group $H^k(G)$ is naturally isomorphic to*

$$\frac{closed\ k-forms\ on\ G}{\{d\eta : \eta\ is\ an\ invariant\ (k-1)-form\ on\ G\}}$$

.

Remark 3.1. *The connected compact Lie group could also be replaced to a G/H, where G is a connected compact Lie group and H is a closed subgroup. Its proof could be found in* [4].

Now we show that t defined in Corollary 2.1 is a nontrivial element of $H^3(G)$.

Theorem 3.1. *Let G be a nonabelian connected compact Lie group. Then $H^3(G)$ is nontrivial.*

Proof. Let $\langle -, - \rangle$ be a bi-invariant metric on G and t be the invariant $3-$form defined in Corollary 2.1. Then for some X, Y that $[X, Y] \neq 0$, $t(X, Y, [X, Y]) = \langle [X, Y], [X, Y] \rangle \neq 0$, so t is nonzero. We have known that t is a closed form. If it equals to $\mathrm{d}\eta$ for some bi-invariant $2-$form η, then $t = \mathrm{d}\eta = 0$, thus t defines a nontrivial cohomology class in $H^3(G)$. □

From this we conclude that if S^n is a Lie group, then $n \in \{0, 1, 3\}$, since $H^3(S^n)$ is nontrivial if and only if $n = 3$. However, is there really a Lie group structure on S^3?

4. $SO(3)$ and $SU(2)$

The special orthogonal group $SO(3)$ is a matrix Lie group, we claim that it is diffeomorphic to the real projective space $\mathbb{R}P^3$.

Theorem 4.1. *$SO(3)$ is diffeomorphic to $\mathbb{R}P^3$.*

Proof. Let D^3 be the closed ball in $\mathbb{R}^3$ centered at the origin and with radius π. Then we could define a map $D^3 \to SO(3)$ by for a vector v in D^3, let the associated rotation in $SO(3)$ have the axis collinear with v and have the angle $||v||$. Obviously if two vectors are mapped to the same rotation, then they are antipodal points on the boundary of D^3. Thus there is a diffeomorphism from $\mathbb{R}P^3$ to $SO(3)$. □

By Theorem 1.3, through $S^3 \to \mathbb{R}P^3$ is a covering, we immediately prove the below corollary.

Corollary 4.1. *There exists a Lie group structure on S^3.*

For instance, we would like to construct the matrix Lie group structure on S^3. In fact,

Theorem 4.2. *S^3 is diffeomorphic to the special unitary group $SU(2)$.*

To prove this theorem, we would like to introduce one other group isomorphic to $SU(2)$, which is correspondent to the quaternions group $\mathbb{H}$.

Definition 4.1. *The group*

$$S\mathbb{H} := \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H} : a, b, c, d \in \mathbb{R}, a^2 + b^2 + c^2 + d^2 = 1\}$$

is called the unit quaternions group, where the group operation is quaternion multiplication (determined by $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$*) and the group inverse is quaternion conjugation.*

We could identify S^3 with $S\mathbb{H}$ by $[(a, b, c, d) \mapsto a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}]$, obviously $S^3 = S\mathbb{H}$ is a Lie subgroup of $\mathbb{H}$. Now we are ready to prove Theorem 4.2.

Proof. By identify the imaginary unit $\mathbf{i} \in \mathbb{C}$ and $\mathbf{i} \in \mathbb{H}$, we could construct a bijection

$$\mathbb{C}^2 \to \mathbb{H}$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto z_1 + \mathbf{j}z_2$$

which induce a map φ from quaternion multiplication to linear map, or precisely,

$$\varphi : \mathbb{H} \to \mathbb{C}^{2\times 2}$$

$$w_1 + \mathbf{j}w_2 \mapsto \begin{pmatrix} w_1 & -\bar{w_2} \\ w_2 & \bar{w_1} \end{pmatrix}$$

If $\gamma \in S\mathbb{H}$, then it is readily to check $\varphi(\gamma) \in SU(2)$. Conversely, if $A \in SU(2)$, then A must have the form

$$\begin{pmatrix} w_1 & -\bar{w_2} \\ w_2 & \bar{w_1} \end{pmatrix}$$

then $\varphi^{-1}(A)$ must be $w_1 + \mathbf{j}w_2$, that shows φ is a bijection between $S\mathbb{H}$ and $SU(2)$.

From the above discussion it is clear that φ and φ^{-1} are smooth maps since their components are linear functions of the coordinates. Since $SU(2)$ and $S\mathbb{H}$ are submanifolds of $\mathbb{C}^{2\times 2}$ and $\mathbb{H}$ respectively, the restrictions of φ and φ^{-1} to these submanifolds are also smooth. Therefore, φ is a diffeomorphism, and so $S^3 = S\mathbb{H}$ is diffeomorphic to $SU(2)$. Moreover, we could check φ is an abstract group homomorphism between $SU(2)$ and $S\mathbb{H}$, hence it is a Lie group isomorphism. $\square$

The last object we would like to construct is the covering homomorphism from $SU(2)$ to $SO(3)$. We now regard $\mathbb{R}^3$ as the linear space spanned by $\mathbf{i}, \mathbf{j}, \mathbf{k} \in \mathbb{H}$, and consider the map

$$\psi : S\mathbb{H} \to [\mathbb{R}^3 \to \mathbb{R}^3]$$

$$q \mapsto [v \mapsto qvq^{-1}].$$

It is easy to check that $[v \mapsto qvq^{-1}] \in SO(3)$, and obviously, $\psi(q) = \psi(q') \iff q = \pm q'$. Moreover, ψ is a homomorphism. At last, we could also write down the matrix version of ψ with respect to the fixed basis of $\mathbb{R}^3$. Explicitly,

Theorem 4.3.

$$\psi : SU(2) \to SO(3)$$

which maps

$$\begin{pmatrix} a + b\mathbf{i} & -c + d\mathbf{i} \\ c + d\mathbf{i} & a - b\mathbf{i} \end{pmatrix} \in SU(2)$$

to

$$\begin{pmatrix} a^2+b^2-c^2-d^2 & 2(bc-ad) & 2(ac+bd) \\ 2(bc+ad) & a^2+c^2-b^2-d^2 & 2(cd-ab) \\ 2(bd-ac) & 2(ab+cd) & a^2+d^2-b^2-c^2 \end{pmatrix} \in SO(3)$$

is a covering homomorphism.

References

[1] Frank, W.W., *Foundations of Differentiable manifolds and Lie Groups*, Springer-Verlag Berlin Heidelberg GmbH, 1983.

[2] John, M. , *Curvatures of Left Invariant Metrics on Lie Groups*, ADVANCES IN MATHEMATICS 21, 1976.

[3] Manfredo, P.C., *Riemannian Geometry*, Birkhäuser, 1992.

[4] Michael, S., *A Comprehensive Introduction to Differential Geometry*, Publish or Perish, 1999.

[5] Wolfgang, Z., *Lie Groups. Representation Theory and Symmetric Spaces*, University of Pennsylvania, 2010.

[6] Thomas, J.M., *The Identity of* $SU(2)$ *and* S^3, http://www.integralscience.org/, 1992.